CHAPTER I

§ 1. Definitions.

Scalar and Vector Quantities.

The mathematicians and physicists have got to deal with two different kinds of quantities. Some of them are specified by a single real number called the magnitude or the measure of that quantity and are not related to any direction in space. Such quantities are **Scalars.** The examples of such quantities are mass, volume, electric charge, temperature, density etc. etc. Thus the mass of a body may be specified by a positive number m equal to the ratio of the given mass to the unit mass.

Thus for scalars we should have a unit of measurement and a real number m that expresses the ratio of the given scalar quantity to that of the unit.

There are however other types of quantities which have got magnitude as well as a definite direction in space. Such type of quantities are called **vector quantities** or simply vectors. The most familiar examples of this type are velocity, acceleration, force, displacement etc. If we say that the speed of a train is 45 m. p. h., our statement is not complete so long as we do not specify the direction in which the train is moving. Similarly we cannot content ourselves by simply giving the magnitude of a force; we have got to specify the particular direction in which it acts. Thus a scalar quantity cannot completely specify a vector quantity.

Representation and Notation of Vectors.

Symbolically a vector is often denoted by two letters with an arrow over them; the tail of the arrow is called the **origin** of the vector whereas its head is called the **terminus** or the end. Thus in the vector $\overrightarrow{AB}$ A is called the origin and B the terminus. The magnitude of the vector is given by the length AB and its direction is from A to B.

Such vectors are called **line vectors.** Thus a vector may be represented by a directed line segment *i.e.* a given portion of a given line on which the two end points origin and the terminus are specified, *i.e.* they cannot be interchanged, for it will change the direction of the vector.

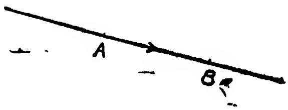


Fig. 1.

In addition to the above notation of vectors by giving their origin and terminus ws shall use sigle letters (Clarendon letters) in heavy (bold face) type like **a, b, c.** The corresponding italic letters a, b, c denote the magnitude of the vectors.

Thus if $\overrightarrow{AB}=\mathbf{a}$, then $|\overrightarrow{AB}|$ *i.e.* its magnitude is denoted by a.

The above notation is all right so far as the printing of the books is concerned but it is quite inconvenient for the teachers and the student to use the above notation on the black-boards and their note-books respectively. Alternatively we may adopt the Greek letters α, β, γ, δ to denote the vectors and the letters a, b, c, d to denote their magnitudes respectively. When we have to deal with large number of

vectors we may use the capital letters A, B, C, D, E, F..for vectors and the corresponding small letters a, b, c, d, e, f... for their magnitudes or by placing bars on the letters *i.e.* by $\bar{a}$, $\bar{b}$, $\bar{c}$ etc. In this book, however, we shall follow the standard notations of using bold-face typed letters for vectors and the corresponding italic letters for their magnitudes.

Free and Localised Vectors.

When we are at liberty to choose the origin of the vector, at any point, then it is said to be a free vector, but when it is restricted to a certain specified point, then the vector is said to be a localized vector.

Equal Vectors. Two vectors are said to be equal when they have the same length (magnitude) and the same direction and the equality of two vectors is written as usual $\mathbf{a}=\mathbf{b}$. Thus equal vectors may be represented by parallel lines of equal length drawn in the same sense of direction irrespective of the origin.

Like Vectors. Vectors are said to be like when they have the same sense of direction.

Collinear Vectors. Any number of vectors are said to be collinear when they are parallel to the same line whatever their magnitudes may be.

Coplanar Vectors. Any number of vectors are said to be coplanar when they are parallel to the same plane.

Co-initial Vectors. A vector is not altered by shifting it about parallel to itself in space. Hence any vector $\mathbf{a}=\overrightarrow{AB}$ may be drawn from any assigned origin O by moving the segment AB parallel to itself so that the point

A coincides with O and the point B falls on some other point say P; then $\overrightarrow{AB}=\overrightarrow{OP}=\mathbf{a}$.

In this way all vectors in space may be replaced by vectors drawn from the same assigned origin by moving them parallel to themselves till their origin coincides with the same assigned origin O. **All such vectors which have the same point as the origin are called co-initial vectors.**

Zero Vector. If the origin and terminal points of a vector coincide, then it is said to be a zero vector. Evidently its length is zero and its direction is indeterminate. A zero vector is denoted by the bold face typed **o.** All zero vectors are equal and they can be expressed as $\overrightarrow{AA}$, $\overrightarrow{BB}$ etc.

Unit Vector. A vector is said to be a unit vector if its **magnitude be of unit length.** If there be any vector **a** whose module is a, then the corresponding unit vector in that direction is denoted by $\hat{\mathbf{a}}$ which has its magnitude unity. Thus we have $\mathbf{a}=a\hat{\mathbf{a}}$ or $\hat{\mathbf{a}}=\dfrac{\mathbf{a}}{a}$.

Note. Any two unit vectors may not be confused to be equal so long as we do not ascertain that their directions are also the same. **Thus only two like unit vectors are equal.** Again if we say that $\mathbf{a}=\mathbf{b}$, it would mean that $\hat{\mathbf{a}}=\hat{\mathbf{b}}$ and that $a=b$, the first relation specifying same direction and the second one specifying equal magnitude.

Position vector. The position vector of any point P with reference to an origin O is the vector $\overrightarrow{OP}$. Thus taking

O as origin we can find the position vector of every point in space. Conversely, corresponding to any given vector $\mathbf{r}$ there is a point P such that $\overrightarrow{OP} = \mathbf{r}$.

Reciprocal vector. A vector whose direction is the same as that of a given vector $\mathbf{a}$ but whose magnitude is reciprocal of the magnitude of the given vector is called the reciprocal of $\mathbf{a}$ and is written as $\mathbf{a}^{-1}$. Thus $\mathbf{a} = a\hat{\mathbf{a}}$.

$$\therefore \quad \text{reciprocal vector } \mathbf{a}^{-1} = \frac{1}{a}\hat{\mathbf{a}} = \frac{a \cdot \hat{\mathbf{a}}}{a^2} = \frac{\mathbf{a}}{a^2}.$$

Now since the magnitude of a unit vector is a unit whose reciprocal is again a unit, we conclude that the **reciprocal of a unit vector is the unit vector itself.**

Negative vector. A vector whose magnitude is the same as that of a given vector $\mathbf{a}$ but opposite direction is called the negative vector of $\mathbf{a}$ and is written as $-\mathbf{a}$. If $\mathbf{a}$ is represented by $\overrightarrow{OA}$ then $-\mathbf{a}$ is represented by $\overrightarrow{AO}$.

§ 2. Addition of vectors.

Let there be any two given vectors $\mathbf{a}$ and $\mathbf{b}$. Now choose any point O and draw the vectors $\mathbf{a}$ and $\mathbf{b}$ so that the terminus of $\mathbf{a}$ coincides with the origin of $\mathbf{b}$, *i.e.* $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$.

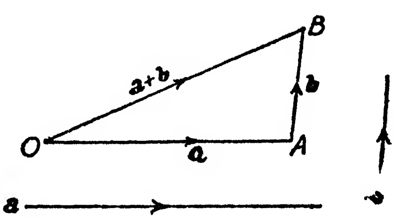


Fig. 2

Then the vector given by $\overrightarrow{OB}$ is defined as the sum of

the given vectors **a** and **b** and is written as

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\mathbf{a}+\mathbf{b}.$$

The above sum is independent of the choice for the position of O. It should be clearly understood here that the magnitude of $\overrightarrow{OB}$ is not equal to sum of the magnitudes of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ as we know that any side of a triangle is less than the sum of the other two. The magnitude of vector OB will only be equal to the sum of the magnitudes of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ when $\overrightarrow{OA}$ and $\overrightarrow{OB}$ have the same direction. The vector $\overrightarrow{OB}$ represents the combined effect (*i.e.* resultant) of vector $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The above law is called **triangle law of addition** by which the vector quantities are compounded.

If $\mathbf{a}+\mathbf{b}=0$ *i.e.* when O and B coincide then $\mathbf{b}=-\mathbf{b}+\mathbf{a}$ showing that $-\mathbf{a}$ is a vector which has the same length as **a** but whose direction is opposite to that of **a**.

Vector addition is Commuative, *i.e.* $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

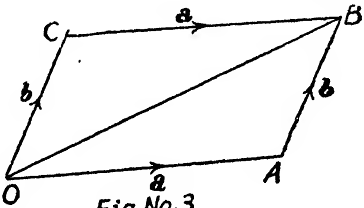


Fig.No.3

If $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{AB}=\mathbf{b}$, then $\overrightarrow{OB}=\mathbf{a}+\mathbf{b}$. Now complete the parallelogram whose two sides are OA and AB. Since the opposite sides of a parallelogram are equal and parallel, we can say that

$$\overrightarrow{OA}=\overrightarrow{CB}=\mathbf{a} \text{ and } \overrightarrow{AB}=\overrightarrow{OC}=\mathbf{b}.$$

$$\therefore \quad \overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}=\overrightarrow{OC}+\overrightarrow{CB}$$

or $$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}.$$

Vector addition is associative

i. e. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$:

Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{AB}=\mathbf{b}$ and $\overrightarrow{OA}=\mathbf{c}$.

Join O to the terminus of last vector C.

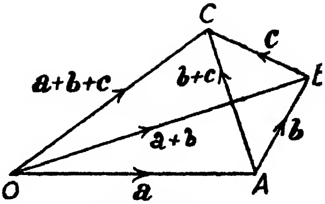


Fig. No. 4

$$\therefore \quad \mathbf{a}+\mathbf{b}=\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}.$$

$$\therefore \quad (\mathbf{a}+\mathbf{b})+\mathbf{c}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}.$$

Similarly $\mathbf{b}+\mathbf{c}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}.$

$$\therefore \quad \mathbf{a}+(\mathbf{b}+\mathbf{c})=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}.$$

Thus we see that $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\overrightarrow{OC}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ where $\overrightarrow{OC}$ is the sum of the three given vectors and is written as $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

Thus the sum of any number of vectors is independent of the order in which they are added and of their grouping to form partial sums.

Sum of any number of vectors. If we are to find the sum of any number of vectors **a**, **b**, **c**, **d**, **e** say, then form a broken line whose segments in length and direction represent these vectors.

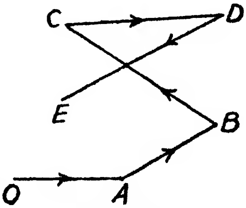


Fig. No. 5

In the above figure $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{\mathbf{AB}}=\mathbf{b}$, $\overrightarrow{BC}=\mathbf{c}$, etc.; then the vector joining the origin of first vector to terminal

point of the last vector will represent the vactor sum **a**+**b**+**c**+**d**+**e** *i. e.* $\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}=\overrightarrow{OE}.$

The above sum will be zero if the terminal point of the last vector coincides with the origin of the first vector.

$$\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DO}=0.$$

From above we observe that

$$\overrightarrow{OA}+\overrightarrow{AA}=\overrightarrow{OA} \text{ or } \overrightarrow{OO}+\overrightarrow{OA}=\overrightarrow{OA}.$$

Now $\overrightarrow{AA}=\mathbf{o}$ and $OO=\mathbf{o}$ and if we put $OA=\mathbf{a}$, then

$$\mathbf{a}+\mathbf{o}=\mathbf{a} \text{ and } \mathbf{o}+\mathbf{a}=\mathbf{a}.$$

Ex. 1. *If* **a, b, c, d** *represent the consecutive sides of a quadrilateral, show that necessary and sufficient condition that the quadrilateral be a parallelogram is that* **a**+**c**=*0 and that this implies* **b**+**d**=*0.*

Since the origin of the first vector coincides with the end point of the last vector, we have

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DA}=0$$

or $$\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0. \;.. \quad (1)$$

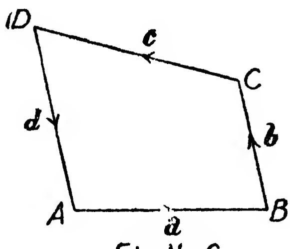


Fig. No. 6

If the figure be a parallelogram, then $AB=CD$ and they are parallel.

$\therefore\ \overrightarrow{AB}=-\overrightarrow{CD}$ as they are in opposite directions or $\mathbf{a}=-\mathbf{c}$ *i. e.* $\mathbf{a}+\mathbf{c}=0$. Hence from (i), we get $\mathbf{b}+\mathbf{d}=0$. $\therefore\ \overrightarrow{BC}=-\overrightarrow{DA}$ giving that BC and DA are equal and parallel. Hence the figure is a parallelogram. We can prove the converse of it easily.

§ 3. Subtraction of vectors.

We have already defined negative vectors as a vector having the same magnitude as that of a given vector but opposite direction. Thus if $\overrightarrow{AB}=\mathbf{b}$, then $\overrightarrow{AC}=-\mathbf{b}$, where $AC=AB$.

The operation of vector subtraction of two given vectors **a** and **b** may be regarded as the operation of addition of vecctor **a** and $-\mathbf{b}$ and written as

$$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b}).$$

Similarly, $\mathbf{b}-\mathbf{a}=\mathbf{b}+(-\mathbf{a})$.

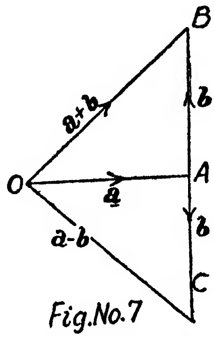


Fig.No.7

Expression of a vector in terms of the position vectors of its end points.

Let us choose any point O as origin and the position vectors of the extremities A and B of a vector $\overrightarrow{AB}$ with respect to this origin be **a** and **b**; then

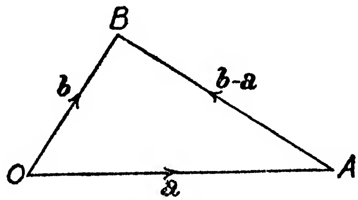


Fig No.8

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{OB}+(-\overrightarrow{OA})$$

$$=\overrightarrow{OB}-\overrightarrow{OA}=\mathbf{b}-\mathbf{a}.$$

Similarly $$\overrightarrow{BA}=\overrightarrow{BO}+\overrightarrow{OA}=\overrightarrow{OA}+(-\overrightarrow{OB})$$

$$=\overrightarrow{OA}-\overrightarrow{OB}=\mathbf{a}-\mathbf{b}.$$

Ex. 2. *If* **a** *and* **b** *are the vectors forming consecutive sides of a regular hexagon ABCDEF, express the vectors* $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FA}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ *and* $\overrightarrow{CE}$ *in terms of* **a** *and* **b**.

(Utkal B. Sc. Hons. 53)

$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\mathbf{a}+\mathbf{b}$ .....(1)

$\overrightarrow{AD}=2\overrightarrow{BC}=2\mathbf{b}$ .....(2)

because $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$ and twice its length.

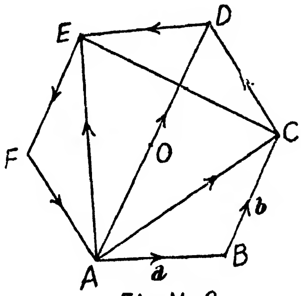


Fig. No. 8

$$\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$$

or $(\mathbf{a}+\mathbf{b})+\overrightarrow{CD}=2\mathbf{b}$. [by (1) & (2)]

$$\therefore\ \overrightarrow{CD}=\mathbf{b}-\mathbf{a}. \qquad \ldots \quad \ldots(3)$$

$$\therefore\ \overrightarrow{FA}=-\overrightarrow{CD}=\mathbf{a}-\mathbf{b}. \qquad \text{[by (3)]}$$

$\therefore$ $\overrightarrow{FA}$ is equal to $\overrightarrow{CD}$ but in opposite direction,

$$\overrightarrow{DE}=-\overrightarrow{AB}=-\mathbf{a} \qquad \ldots \quad \ldots(4)$$

$$\therefore\ \overrightarrow{CE}=\overrightarrow{CD}+\overrightarrow{DE}=\mathbf{b}-\mathbf{a}-\mathbf{a}=\mathbf{b}-2\mathbf{a}, \qquad \text{[by (3) and (4)]}$$

$$\overrightarrow{EF}=-\overrightarrow{BC}=-\mathbf{b},$$

$$\overrightarrow{AE}=\overrightarrow{AD}+\overrightarrow{DE}=2\mathbf{b}-\mathbf{a}. \qquad \text{[by (2) and (4)]}$$

Ex. 3. *The position vectors of four points A, B, C, D are*

a, b, 2a+3b, a − 2b *respectively. Express the vectors* $\overrightarrow{AC}$, $\overrightarrow{DB}$, $\overrightarrow{BC}$ *and* $\overrightarrow{CA}$ *in terms of* **a** *and* **b.**

Let O be the origin so that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=2\mathbf{a}+3\mathbf{b}$ and $\overrightarrow{OD}=\mathbf{a}-2\mathbf{b}$.

$$\therefore \quad \overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(2\mathbf{a}+3\mathbf{b})-\mathbf{a}=\mathbf{a}+3\mathbf{b},$$

$$\overrightarrow{DB}=\overrightarrow{OB}-\overrightarrow{OD}=\mathbf{b}-(\mathbf{a}-2\mathbf{b})=3\mathbf{b}-\mathbf{a},$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(2\mathbf{a}+3\mathbf{b})-\mathbf{b}=2\,(\mathbf{a}+\mathbf{b}),$$

$$\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=\mathbf{a}-(2\mathbf{a}+3\mathbf{b})=-(\mathbf{a}+3\mathbf{b}).$$

Ex. 4. *Five forces* $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$ *act at the vertex A of a regular hexagon ABCDEF. Prove that the resultant is* $6\overrightarrow{AO}$ *where O is the centroid of the hexagon.*

Refer figure Ex. 2. If R be the resultant, then

$$R=\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$$

$$=\overrightarrow{AB}+(\overrightarrow{AD}+\overrightarrow{DC})+\overrightarrow{AD}+(\overrightarrow{AD}+\overrightarrow{DE})+\overrightarrow{AF}.$$

Now $\overrightarrow{AF}$ and $\overrightarrow{DC}$ are two equal vectors in opposite directions and hence they cancel each other. Similarly $\overrightarrow{AB}$ and $\overrightarrow{DE}$ cancel each other.

$\therefore$ $R=3\overrightarrow{AD}=6.\overrightarrow{AO}$ where O is the mid. point of AD *i. e.* the centroid of the hexagon.

§ 4. Multiplication of vectors by scalars.

If $\overrightarrow{OA}=\mathbf{a}$ and m is any positive real number then the vector $m.\overrightarrow{OA}=m\mathbf{a}$ is defined to be a vector in the direction of the given vector but whose module is m times the module of the given vector.

In a similar manner the vector $-m.\overrightarrow{OA}=-m\mathbf{a}$ is a vector in the direction opposite to that of $\mathbf{a}$ and having module m times that of $\mathbf{a}$.

The division of a vector by a real number m may be considered as the product of that vector by $\frac{1}{m}$.

Multiplication of vectors by scalars is commutative,,associative and distributive, i. e.

$$m.\mathbf{a}=\mathbf{a}.m,$$

$$m\,(n\mathbf{a})=(mn)\,\mathbf{a}=n\,(m\mathbf{a}),$$

$$(m+n)\,\mathbf{a}=m\mathbf{a}+n\mathbf{a},$$

$$m\,(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b},$$

where m and n are any scalar numbers and $\mathbf{a}$ and $\mathbf{b}$ are any two vectors.

The first three results follow from the definition and we are going to prove here the last result.

Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{AB}=\mathbf{b}$ so that $\overrightarrow{OB}=\mathbf{a}+\mathbf{b}$. If m be a +ive number, then choose A' and B' on OA and OB produced respectively, so that $OA'=m.OA$ and $OB'=m.OB$.

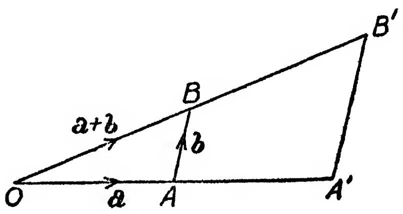


Fig. No. 10

$\therefore \quad \overrightarrow{OA'} = m.\mathbf{a},\ \overrightarrow{OB'} = m.(\mathbf{a}+\mathbf{b}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$

Since $A'B'$ is parallel to AB, $\quad \therefore \quad A'B' = m . AB$.

$\therefore \quad \overrightarrow{A'B'} = m . \overrightarrow{AB}$ or $\overrightarrow{A'B'} = m . \mathbf{b} \ldots \quad \ldots \quad \ldots \quad \ldots(2)$

Now $\qquad \overrightarrow{OA'} + \overrightarrow{A'B'} = \overrightarrow{OB'}$

or $\qquad m . \overrightarrow{OA} + m . \overrightarrow{AB} = m . \overrightarrow{OB}$

or $\qquad m . \mathbf{a} + m . \mathbf{b} = m(\mathbf{a}+\mathbf{b})$. $\quad$ [from (1) and (2)]

Note—In case m be negative then we should choose a point A' on AO produced (and not OA produced) such that OA' is m times OA but in direction opposite to that of OA. The above result can be similarly proved by the help of the diagram given.

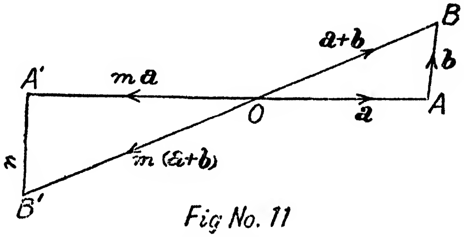


Fig No. 11

Any of two like vectors can be expressed as a multiple of the other.

Let $\overrightarrow{AB} = \mathbf{a} = a . \hat{\mathbf{a}}$ and $\overrightarrow{CD} = \mathbf{b} = b . \hat{\mathbf{b}}$

be two like vectors so that $\hat{\mathbf{a}} = \hat{\mathbf{b}} \ldots(1)$

$\therefore \quad \mathbf{b} = b . \hat{\mathbf{b}} = \frac{b}{a} . a\hat{\mathbf{b}} = \frac{b}{a} . a\hat{\mathbf{a}}$

$= \frac{b}{a} . \mathbf{a}$ from (1).

Thus $\overrightarrow{CD} = \frac{b}{a} . \overrightarrow{AB}$.

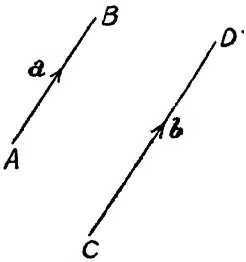


Fig. No. 12

Hence proved.

Similarly we can show that $\mathbf{a}=\frac{a}{b}\,.\,\mathbf{b}$ *i. e.* $\overrightarrow{AB}=\frac{a}{b}\overrightarrow{CD}$.

§ 5. Linear Combination of Vectors.

If a vector $\mathbf{r}$ can be expressed as

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+\ldots$$

where $x, y, z\ldots$ are any scalar numbers, then r is said to be a linear combination of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}\ldots$

§ 6. Components of Vectors.

We have already defined that when any number of vectors are parallel to the same line, they are said to be collinear and when they are parallel to the same plane they are said to be coplanar.

(i) Collinear Vector :—*Any vector* $\mathbf{r}$ *collinear with a given vector* $\mathbf{a}$ *can be expressed as* $x\mathbf{a}$ *where* x *is a scalar.*

Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OP}=\mathbf{r}$.

Since $\overrightarrow{OP}$ is collinear with $\overrightarrow{OA}$, it can be expressed as a scalar multiple of OA, *i. e.* $\overrightarrow{OP}=x\,.\,\overrightarrow{OA}$

or $$\mathbf{r}=x.\mathbf{a}.$$

Coplanar vectors :—*Any vector* $\mathbf{r}$ **coplanar** *with any two given* **(non-collinear)** *vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *can be* **uniquely** *expressed as a linear combination of the given vectors i. e.* $\mathbf{r}=x\mathbf{a}+y\mathbf{b}$ *where* x *and* y *are scalars.* **(Raj. B. Sc. 1960)**

Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$ be two non-collinear vectors and $\overrightarrow{OP}=\mathbf{r}$ be a vector coplanar with $\mathbf{a}$ and $\mathbf{b}$ Now through P draw PM and PN parallel to OB and OA respectively and meeting OA in M and OB in N.

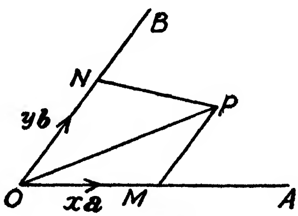


Fig. No. 13.

$\therefore$ $\overrightarrow{OM}$ being collinear with $\overrightarrow{OA}=x\mathbf{a}$,

$\overrightarrow{ON}$ being collinear with $\overrightarrow{OB}=y\mathbf{b}$.

$$\therefore \quad \mathbf{r}=\overrightarrow{OP}=\overrightarrow{OM}+\overrightarrow{MP}=\overrightarrow{OM}+\overrightarrow{ON}=x\mathbf{a}+y\mathbf{b}.$$

Hence proved.

The above combination is unique.

In order to prove that the linear combination

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b} \qquad \ldots \quad \ldots(1)$$

is unique, let us suppose that r can be expressed in another form as

$$\mathbf{r}=x'\mathbf{a}+y'\mathbf{b} \qquad \ldots. \quad (2)$$

$$\therefore \quad x\mathbf{a}+y\mathbf{b}=x'\mathbf{a}+y'\mathbf{b}. \quad \text{[from (1) and (2)]}$$

$$\therefore \quad (x-x')\,\mathbf{a}+(y-y')\,\mathbf{b}=0$$

or
$$p\mathbf{a}+q\mathbf{b}=0.$$

$$\therefore \quad \mathbf{a}=-\frac{q}{p}\,\mathbf{b}.$$

If p be not equal to zero, then **a** is a scalar multiple of **b**.

The above form shows that **a** and **b** are collinear which contradicts that **a** and **b** are non-collinear. Hence we must have $p=0$ or $x-x'=0$ or $x=x'$.

Similarly by writing $\mathbf{b}=-\dfrac{p}{q}\,\mathbf{a}$ we can prove that $q=0$ or $y-y'=0$ or $y=y'$. Hence we prove that the above combination is unique.

Therefore if two equal vectors are expressed in terms of the same two non-collinear vectors **a** *and* **b**, *the corresponding scalar coefficients are equal.*

The above result will not be true in case the two vectors **a** and **b** be collinear [see note after case (iii).]

Note :—*From above we also observe that if there exists a*

relation of the form $p\mathbf{a}+q\mathbf{b}=0$ *between two non-collinear vectors* $\mathbf{a}$ *and* $\mathbf{b}$, *then* $p=0$, $q=0$ *(p and q being scalars).*

If there be several vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3 \ldots$ coplanar with two non-collinear vectors $\mathbf{a}$ and $\mathbf{b}$, then they may be expressed as

$$\mathbf{r}_1 = x_1\mathbf{a} + y_1\mathbf{b},$$
$$\mathbf{r}_2 = x_2\mathbf{a} + y_2\mathbf{b},$$
$$\mathbf{r}_3 = x_3\mathbf{a} + y_3\mathbf{b}.$$

If $\mathbf{r}$ be their sum, then
$$\mathbf{r} = (\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\ldots) = (x_1+x_2+x_3+\ldots)\,\mathbf{a} + (y_1+y_2+y_3+\ldots)\,\mathbf{b}.$$

Above relation shows that the components of a sum of vectors are the sums of components of these vectors. In case $\mathbf{r}=0$ then each of its components must be zero (as $\mathbf{a}$ and $\mathbf{b}$ are non-collinear). Therefore the vector equation $\mathbf{r}=0$ is equivalent to the two scalar equations.

$$x_1+x_2+x_3+\ldots=0 \text{ and } y_1+y_2+y_3+\ldots=0.$$

Ex. 5. *Prove that the following vectors are coplanar*

$$3\mathbf{a}-7\mathbf{b}-4\mathbf{c},\ 3\mathbf{a}-2\mathbf{b}+\mathbf{c},\ \mathbf{a}+\mathbf{b}+2\mathbf{c},$$

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *being any vectors.*

If these vectors are coplanar, we should be able to express one of them as a linear combination of the other two. Let us suppose that the given vectors are coplanar.

$\therefore$ $3\mathbf{a}-7\mathbf{b}-4\mathbf{c} = x\,(3\mathbf{a}-2\mathbf{b}+\mathbf{c}) + y\,(\mathbf{a}+\mathbf{b}+2\mathbf{c})$, where x and y are scalars. Comparing the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we get $3x+y=3$, $-2x+y=-7$, $x+2y=-4$. Solving the first two, we find that $x=2$, and $y=-3$. These values of x and y satisfy the third equation as well. Hence the given vectors are coplanar.

(iii) Non-coplanar vectors. *Any vector* $\mathbf{r}$ *can be uniquely expressed as a linear combination of three given (non-*

coplanar) vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *i.e.* $\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$ *where* x, y *and* z *are scalars.* **(Pb. 60)**

Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$ be any three non-coplanar vectors and let $\overrightarrow{OP}=\mathbf{r}$.

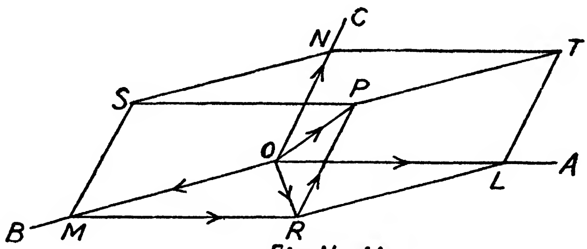


Fig. No. 14

The three lines OA, OB, OC taken in pairs determine three planes BOC, COA and AOB. Now through P draw planes parallel respectively to the planes BOC, COA and AOB cutting OA, OB and OC in L, M and N respectively thus giving us a parallelopiped whose diagonal is OP.

Also $\overrightarrow{OL}$ is collinear with $\overrightarrow{OA}$ *i.e.* $\mathbf{a}$; $\therefore$ $\overrightarrow{OL}=x\mathbf{a}$.

Similarly $\overrightarrow{OM}=y\mathbf{b}$ and $\overrightarrow{ON}=z\mathbf{c}$.

Now $\mathbf{r}=\overrightarrow{OP}=\overrightarrow{OR}+\overrightarrow{RP}=(\overrightarrow{OM}+\overrightarrow{MR})+\overrightarrow{RP}$

$=\overrightarrow{OM}+\overrightarrow{OL}+\overrightarrow{ON}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$.

The above combination is unique.

In order to prove that the linear combination

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}, \qquad \ldots \quad \ldots(1)$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are **non-coplanar** vectors, is unique, let us suppose that $\mathbf{r}$ can be expressed in another form as

$$\mathbf{r}=x'\mathbf{a}+y'\mathbf{b}+z'\mathbf{c} \qquad \ldots \quad \ldots(2)$$

$$\therefore \quad x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=x'\mathbf{a}+y'\mathbf{b}+z'\mathbf{c}. \text{ [from (1) and (2)]}$$

$$\therefore \quad (x-x')\,\mathbf{a}+(y-y')\,\mathbf{b}+(z-z')\,\mathbf{c}=0$$

or
$$L\mathbf{a}+M\mathbf{b}+N\mathbf{c}=0.$$

If L be not equal to zero, then $\mathbf{a}=-\dfrac{M}{L}\,\mathbf{b}-\dfrac{N}{L}\,\mathbf{c}.$

Above shows that **a** can be expressed as a linear combination of two non-collinear vectors **b** and **c**. But it is essential here that **a, b** and **c** should be coplanar [§ 5 P. 14] and we are given that **a, b** and **c** are non-coplanar. Hence there is a contradiction. So we must have

$$L=0 \quad \text{or} \quad x-x'=0, \quad i.e. \ x=x'.$$

Similarly $M=0$ or $y-y'=0$, *i.e.* $y=y'$

and $N=0$ or $z-z'=0$, *i.e.* $z=z'$.

Hence we prove that the above combination is unique.

Therefore if two equal vectors are expressed in terms of the same three non-coplanar vectors, the corresponding scalar coefficients are equal.

It should be understood here that the above result will not necessarily be true when **a, b** and **c** are coplanar as in that case **c** can be expressed in terms of **a** and **b**,

i.e.,
$$\mathbf{c}=p\mathbf{a}+q\mathbf{b}.$$

$$\therefore \quad \mathbf{r}_1=x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c}=(x_1+pz_1)\,\mathbf{a}+(y_1+qz_1)\,\mathbf{b},$$

$$\mathbf{r}_2=x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c}=(x_2+pz_2)\,\mathbf{a}+(y_2+qz_2)\,\mathbf{b}.$$

$$\therefore \quad \mathbf{r}_1-\mathbf{r}_2=\{(x_1+pz_1)-(x_2+pz_2)\}\,\mathbf{a}+\{(y_1+qz_1)-(y_2+qz_2)\}\,\mathbf{b}=0$$

$\mathbf{r}_1$ and $\mathbf{r}_2$ are equal.

$\therefore$ since **a** and **b** are two vectors supposed non-collinear, we must have their coefficients each zero.

$$\therefore \quad x_1+pz_1=x_2+pz_2$$

and
$$y_1+qz_1=y_2+qz_2.$$

The above relations do not necessarily mean that

$$x_1=x_2,\ y_1=y_2 \text{ and } z_1=z_2.$$

Note :—From above we also observe that if **a, b, c** are three non-copalanar vectors and there exists a relation of the form

$$L\mathbf{a}+M\mathbf{b}+N\mathbf{c}=0 \qquad (L, M, N \text{ being scalar})$$

then $\qquad L=0, M=0, N=0.$

Note :—In the relation $\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$, the vectors $x\mathbf{a}$, $y\mathbf{b}$, $z\mathbf{c}$ are called the **components** of the vector $\mathbf{r}$ and x, y, z are called the **coordinates of the point P** with reference to the vectors **a, b, c**.

If there be several vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \ldots$ expressed in terms of three non-coplanar vectors **a, b, c**, then they may be expressed as

$$\mathbf{r}_1=x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c},$$
$$\mathbf{r}_2=x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c},$$
$$\mathbf{r}_3=x_3\mathbf{a}+y_3\mathbf{b}+z_3\mathbf{c}.$$

If $\mathbf{r}$ be their sum, then

$$\mathbf{r}=(\Sigma\, x_1)\, \mathbf{a}+(\Sigma\, y_1)\, \mathbf{b}+(\Sigma\, z_1)\, \mathbf{c}.$$

Above relation shows that the components of a sum of vectors are the sums of the components of those vectors. In case $\mathbf{r}=0$, then each of its components should be zero (as **a, b, c** are three non-coplanar vectors). Hence the vector equation $\mathbf{r}=0$ is equivalent to three scalar equations.

$$\Sigma\, x_1=0,\ \Sigma\, y_1=0,\ \Sigma\, z_1=0.$$

Ex. 6. *If* **a, b, c** *be any three non-zero, non-coplanar vectors, find the linear relation between the following system of vectors :—*

$7\mathbf{a}+6\mathbf{c},\ \mathbf{a}+\mathbf{b}+\mathbf{c},\ 2\mathbf{a}-\mathbf{b}+\mathbf{c},\ \mathbf{a}-\mathbf{b}-\mathbf{c}.$

Let $7\mathbf{a}+6\mathbf{c}=x\,(\mathbf{a}+\mathbf{b}+\mathbf{c})+y\,(2\mathbf{a}-\mathbf{b}+\mathbf{c})+z\,(\mathbf{a}-\mathbf{b}-\mathbf{c})$.

Comparing the coefficients of **a**, **b** and **c** we, find that

$$x+2y+z=7,\ x-y-z=0,\ x+y-z=6.$$

Solving these three equations we find that $x=2, y=3, z=-1$.

Hence the required linear relation is

$$7\mathbf{a}+6\mathbf{c}=2\,(\mathbf{a}+\mathbf{b}+\mathbf{c})+3\,(2\mathbf{a}-\mathbf{b}+\mathbf{c})-(\mathbf{a}-\mathbf{b}-\mathbf{c}).$$

§ 6. Linear dependence of vectors.

If there exists a relation of the type

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+\ldots=0 \qquad \ldots \quad \ldots \quad \ldots(1)$$

where $x, y, z \ldots\ldots$ are scalars (not all zero), then the system of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots$ is said to be **linearly dependent.**

If the system of vectors is not linearly dependent, then it is said to be linearly independent and in that case

$$x=0, y=0, z=0\ldots$$

If $$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+\ldots$$

i. e. $-\mathbf{r}+x\mathbf{a}+y\mathbf{b}+z\mathbf{c}\ldots=0$, then the system of vectors $\mathbf{r}, \mathbf{a}, \mathbf{b}, \mathbf{c} \ldots.$ is linearly dependent.

Now we know that between two collinear vectors there exists a relation of the form $\mathbf{r}=x\mathbf{a}$. Similarly between three coplanar vectors there exists a relation of the form $\mathbf{r}=x\mathbf{a}+y\mathbf{b}$. Also between any four vectors there exists a relation of the type $\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$.

Therefore we can say that any two collinear vectors, or three coplanar vectors or any four or more vectors form a linearly dependent system.

Combining the results proved in § 5 and § 6, we can say that

The necessary and sufficient condition that two vectors be linearly independent is that they be collinear.

and

The necessary and sufficient condition that three vectors be linearly independent is that they be coplanar.

§ 7. The Unit Vectors i, j, k.

In § 5 case (iii), we expressed any vector $\mathbf{r}$ in terms of three non-coplanar vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ as $\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$.

Here in this case the directions of the three non-coplanar vectors will be mutually perpendicular say OX, OY and OZ and unit vectors in these directions are denoted by **i, j, k** respectively. Such a system of vectors is called **ortho-normal system.** Now we shall express any vector **r** in terms of the three unit vectors **i, j** and **k.**

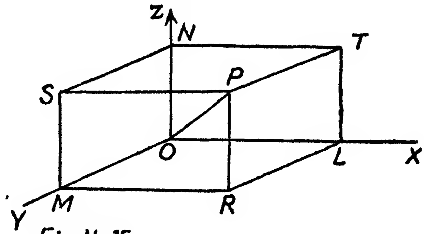


Fig.No.15

Proceeding exactly as in § 5, case (iii), if $OL=x$, $OM=y$, $ON=z$, then

$$\overrightarrow{OP}=\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \qquad \ldots \quad \ldots \quad \ldots(1)$$

x, y and z are called the coordinates of the point P.

Also $OP^2=OL^2+LP^2$ as OL is the orthogonal projection of OP on OX, *i. e.* $\angle OLP=\pi/2$

or $\qquad OP^2=OL^2+PR^2+RL^2=OL^2+ON^2+OM^2$

or $\qquad r^2=x^2+y^2+z^2$, where **r** is the module of $\overrightarrow{OP}$.

i. e. square of the module of vector r is sum of the squares of the coefficients of **i, j** *and* **k** *when* **r** *is expressed in terms of* **i, j, k.**

Direction cosines of OP.

Again if OP makes an angle α with the direction of **i,** then $\cos\alpha=\dfrac{OL}{OP}=\dfrac{x}{r}$.

$\therefore \quad x=r\cos\alpha$. Similarly $y=r\cos\beta$, $z=r\cos\gamma$, where β and γ are the angles which OP makes with the directions of unit vectors **j** and **k** respectively. **cos α, cos β, cos γ are**

called the directions cosines of OP and are written as l, m, n respectively.

$$x^2+y^2+z^2=r^2\,(\cos^2\alpha+\cos^2\beta+\cos^2\gamma).$$

But $x^2+y^2+z^2=r^2$; $\therefore\quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$

Direction ratios of OP.

Also $\cos\alpha=\dfrac{x}{r}$, $\cos\beta=\dfrac{y}{r}$, $\cos\gamma=\dfrac{z}{r}$, showing that the direction cosines of OP are proportional to $x, y,$ and z, *i. e.* the coefficients of **i**, **j** and K and the actual direction cosines are obtained by dividing their coefficients by r, the module of OP; *i. e.* $\sqrt{(x^2+y^2+z^2)}$. x, y, z are called the rectangular coordinates of P.

Note :—In the case of unit vector, the module is unity and hence if a unit vector be resolved in terms of **i, j, k**, then their coefficients themselves are the direction cosines.

§ 8. Distance between two points P_1 and P_2 and the direction cosines of the line joining them.

Choose any point O as origin; then the position vectors of P_1 and P_2 are $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. We can express them in terms of unit vectors **i, j** and **k** as

$$\overrightarrow{OP_1}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k},$$

$$\overrightarrow{OP_2}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k},$$

where $(x_1, y_1\ z_1)$ and (x_2, y_2, z_2) are the rectangular Cartesian coordinates of P_1 and P_2.

$$\therefore\quad \overrightarrow{P_1P_2}=\overrightarrow{OP_2}-\overrightarrow{OP_1}=(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}+(z_2-z_1)\,\mathbf{k}.$$

If r be the distance P_1P_2, then it is the module of $\overrightarrow{P_1P_2}$ and is equal to square root of the sum of the squares of the coefficients of **i**, **j** and **k** and is therefore equal to

$$\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}}.$$

Also the direction ratios of the line P_1P_2 are the the coefficients x_2-x_1, y_2-y_1 and z_2-z_1 of **i**, **j** and **k** recpectively. The actual direction cosines are obtained by dividing these coefficients by the module of $\overrightarrow{P_1P_2}$ *i.e.* r

or $$\sqrt{[\Sigma\ (x_2-x_1)^2]}$$

and are therefore $\dfrac{x_2-x_1}{r}, \dfrac{y_2-y_1}{r}, \dfrac{z_2-z_1}{r}$.

Note :—In case $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are unit vectors, then

$$x_1^2+y_1^2+z_1^2=1,\ x_2^2+y_2^2+z_2^2=1.$$

$$\therefore\ r=\{(x_1^2+y_1^2+z_1^2)\ (x_2^2+y_2^2+z_2^2)-2\ (x_1x_2+y_1\,y_2+z_1z_2)\}^{1/2}$$
$$=\{2-2\ (x_1x_2+y_1\,y_2+z_1z_2)\}^{1/2}.$$

Putting the value of r, we find the corresponding direction cosines.

Ex. 7. *Three vectors of magnitudes a, 2a, 3a meet in a point and their directions are along the diagonals of three adjacent faces of a cube. Determine their relultant R and its direction cosines.*

(Lucknow B. Sc. 51, Utkal B. Sc. Hon's 53, B. H. U. M. Sc. 54)

Prove also that the sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through the same corner, the vectors being directed from that corner is twice the vector determined by the diagonal of the cube.

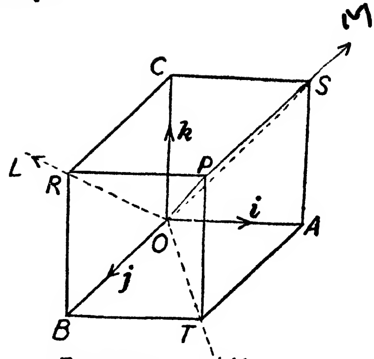


Fig. No. 16

Let the edge of a cube be taken as unity and the vectors represented by OA, OB, OC the three coterminous edges of unit length be $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively. OR, OS and OT are the three diagonals of the three adjacent faces of the cube along which act the vectors $\vec{OL}$, $\vec{OM}$ and $\vec{ON}$ of magnitudes a, $2a$ and $3a$ respectively. In order to find these vectors we shall first find unit vectors in these directions and then multiply them by their corresponding given magnitudes.

$\vec{OR}=\mathbf{j}+\mathbf{k}$ and its module is clearly $\sqrt{(0+1+1)}=\sqrt{2}$.

$\therefore$ unit vector along $OR=\dfrac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})$

Now the magnitude of vector $\vec{OL}$ is a.

$$\therefore \quad \vec{OL}=\frac{a}{\sqrt{2}}(\mathbf{j}+\mathbf{k})$$

Exactly in a similar manner the vector $\vec{OM}$ of magnitude $2a$ is $\dfrac{2a}{\sqrt{2}}(\mathbf{k}+\mathbf{i})$ and vector $\vec{ON}$ of magnitude $3a$ is

$$\frac{3a}{\sqrt{2}}(\mathbf{i}+\mathbf{j}).$$

$$\therefore \quad \vec{OL}+\vec{OM}+\vec{ON}=\frac{a}{\sqrt{2}}(\mathbf{j}+\mathbf{k})+\frac{2a}{\sqrt{2}}(\mathbf{k}+\mathbf{i})+\frac{3a}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$

$$=\frac{5a}{\sqrt{2}}\mathbf{i}+\frac{4a}{\sqrt{2}}\mathbf{j}+\frac{3a}{\sqrt{2}}\mathbf{k}=R.$$

The magnitude of the resultant is

$$r=\sqrt{\left(\frac{25a^2}{2}+\frac{16a^2}{2}+\frac{9a^2}{2}\right)}=5a.$$

$\therefore$ direction cosines are $\frac{4a}{r.\sqrt{2}}, \frac{5a}{r.\sqrt{2}}, \frac{3a}{r.\sqrt{2}}$

or $$\frac{1}{\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}.$$

Again $\overrightarrow{OR}+\overrightarrow{OS}+\overrightarrow{OT}=2\,(\mathbf{i}+\mathbf{j}+\mathbf{k})=2.\overrightarrow{OP}$ which proves second past [on putting $x=y=z=1$ in (1) of § 7 P. 21].

Exercise No. 1

1. **a, b** *are the position vectors of A and B respectively; find that of a point C in AC produced such that $AC=3AB$, and that of a point D in BA produced such that $BD=2BA$.*

If O be the origin, then $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$. Now from $\overrightarrow{AC}=3\overrightarrow{AB}$, we get

$$\overrightarrow{OC}-\overrightarrow{OA}=3\,(\overrightarrow{OB}-\overrightarrow{OA}),\ \therefore\ \overrightarrow{OC}=3\mathbf{b}-2\mathbf{a}.$$

Similarly $\overrightarrow{OD}=2\mathbf{a}-\mathbf{b}$.

2. *If the vertices of a triangle are the points*

$$a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},\ c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k},$$

what are the vectors determined by the sides ? Find the length of these vectors.

See § 8. $(b_1-a_1)\,\mathbf{i}+(b_2-a_2)\,\mathbf{j}+(b_3-a_3)\,\mathbf{k}$ etc.

Sides are $\{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2\}^{1/2}$ etc.

3. *If the postiion vectors of A and B are* $\mathbf{i}+3\mathbf{j}-7\mathbf{k}$ *and* $5\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ *respectively, find* $\overrightarrow{AB}$ *and determine its direction cosines and its module.*

Ans. $4\mathbf{i}-5\mathbf{j}+11\mathbf{k}$, module$=9\sqrt{2}$.

d c.'s are $\dfrac{4}{9\sqrt{2}}, \dfrac{-5}{9\sqrt{2}}, \dfrac{11}{9\sqrt{2}}$.

4. *The position vectors of the points P, Q, R, S are*

$$(\mathbf{i}+\mathbf{j}+\mathbf{k}),\ (2\mathbf{i}+5\mathbf{j}),\ (3\mathbf{i}+2\mathbf{j}-3\mathbf{k}),\ (\mathbf{i}-6\mathbf{j}-\mathbf{k}).$$

Prove that the lines PQ and RS are parallel, and find the ratio of these lengths.

In this question show that the direstion ratios of the lines PQ and RS are proportional. Ratio of their lengths is $\frac{1}{2}$.

5. (a) *In the adjoining figure if* $\vec{OA}=\mathbf{a}$, $\vec{OB}=\mathbf{b}$, $\vec{OC}=\mathbf{c}$, *find the vectors given by* $\mathbf{a}+\mathbf{b}+\mathbf{c}$, $\mathbf{b}+\mathbf{c}-\mathbf{a}$, $\mathbf{c}+\mathbf{a}-\mathbf{b}$, $\mathbf{a}+\mathbf{b}-\mathbf{c}$, *and verify that the four vectors along the four diagonals are linearly independent. Also find the vectors along the diagonals of the faces in terms of* **a, b, c**.

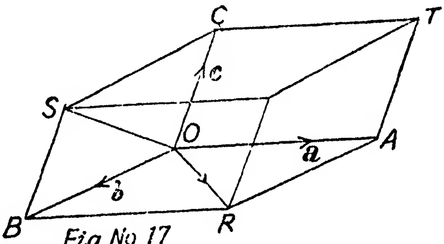


Fig. No 17

$$\vec{OS}=\mathbf{b}+\mathbf{c},\quad \vec{AO}=-\mathbf{a};\ \therefore\ \vec{AS}=\vec{AO}+\vec{OS}=\mathbf{b}+\mathbf{c}-\mathbf{a}\ldots(1)$$

$$\vec{OT}=\mathbf{c}+\mathbf{a},\quad \vec{BO}=-\mathbf{b};\ \therefore\ \vec{BT}=\vec{BO}+\vec{OT}=\mathbf{c}+\mathbf{a}-\mathbf{b}\ldots(2)$$

$$\vec{OR}=\mathbf{a}+\mathbf{b},\quad \vec{CO}=-\mathbf{c};\ \therefore\ \vec{CR}=\vec{CO}+\vec{OR}=\mathbf{a}+\mathbf{b}-\mathbf{c}\ldots(3)$$

$$\vec{OP}=\vec{OR}+\vec{RP}=\vec{OR}+\vec{OC}=\mathbf{a}+\mathbf{b}+\mathbf{c}\quad\ldots\ \ldots\ \ldots(4)$$

From (1), (2), (3) and (4), we find that

$$\vec{OP}=\vec{AS}+\vec{BT}+\vec{CR}.$$

Hence they are linearly independent.

$$\vec{AP}=\vec{OS}=\mathbf{b}+\mathbf{c},\ \vec{TR}=\vec{CB}=\vec{OB}-\vec{OC}=\mathbf{b}-\mathbf{c}.$$

Similarly we can find the other diagonals of the other faces.

(b) *A particle at the corner of a cube is acted on by forces 1, 2, 3 lbs.-wt. respectively along the diagonals of the faces of the cube which meet the particle. Find their resultant.*

Putting $a=1$ in solved Ex. 7 P. 23, we get $R=5$. A force represented by $\frac{1}{\sqrt{2}}(5\mathbf{i}+4\mathbf{j}+3\mathbf{k})$.

6. *ABCD is a quadrilateral. Forces $\vec{BA}, \vec{BC}, \vec{CD}, \vec{DA}$ act at a point. Show that their resultant is $2\vec{BA}$.*

$$R=\vec{BA}+\vec{BC}+\vec{CD}+\vec{DA}. \text{ Add } \vec{AB}+\vec{BA} \text{ i. e. } 0$$

$$=2\vec{BA}+(\vec{AB}+\vec{BC}+\vec{CD}+\vec{DA})=2\vec{BA}+0=2\vec{BA}.$$

7. *ABCDE is a pentagon. Forces $\vec{AB}, \vec{AE}, \vec{BC}, \vec{DC}, \vec{ED}$ and $\vec{AC}$ act at a point. Prove that their resultant is $3\vec{AC}$.*

8. *Find the position vectors of the mid-points of the sides of a triangle in terms of the position vectors of the vertices and hence prove that sum of the vectors determined by the medians of a triangle directed from the vertices is zero.* **(Agra 37)**

$$\vec{BC}=\vec{OC}-\vec{OB}$$

$$=\mathbf{c}-\mathbf{b}.$$

$\therefore\ \ BD=\frac{1}{2}(\mathbf{c}-\mathbf{b})$, D being the mid point of BC.

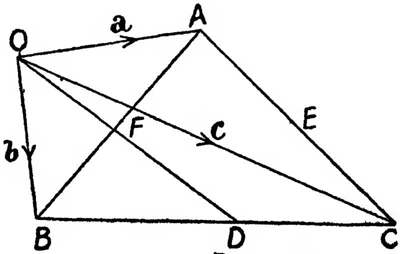


Fig.No.18

$$\therefore\ \ \vec{OD}=\vec{OB}+\vec{BD}$$

$$=\mathbf{b}+\frac{\mathbf{c}-\mathbf{b}}{2}=\frac{\mathbf{b}+\mathbf{c}}{2} \text{ etc.}$$

$$\vec{AD}=\vec{OD}-\vec{OA}=\frac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}=\frac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{2} \text{ etc.}$$

9. *Prove that the following vectors are coplanar :*

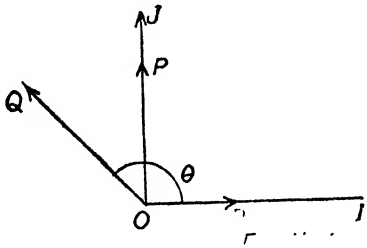
$$5\mathbf{a}+6\mathbf{b}+7\mathbf{c},\ 7\mathbf{a}-8\mathbf{b}+9\mathbf{c},\ 3\mathbf{a}+20\mathbf{b}+5\mathbf{c}.$$

10. *Find the linear relation between the following systems of vectors,* **a**, **b**, **c** *being any three non-coplanar vectors :*

$$\mathbf{a}+3\mathbf{b}+4\mathbf{c},\ \mathbf{a}-2\mathbf{b}+3\mathbf{c},\ \mathbf{a}+5\mathbf{b}-2\mathbf{c},\ 6\mathbf{a}+14\mathbf{b}+4\mathbf{c}.$$

Ans. $6\mathbf{a}+14\mathbf{b}+4\mathbf{c}=(\mathbf{a}+3\mathbf{b}+4\mathbf{c})+2\ (\mathbf{a}-2\mathbf{b}+3\mathbf{c})$
$+3\ (\mathbf{a}+5\mathbf{b}-2\mathbf{c})$

11. *If the resultant of two forces is equal in magnitude to one of the components and perpendicular to it in direction, find the other component.*



Let the force P be horizontal and Q be inclined to it at an angle θ, so that the resultant is a vertical force P. If the unit vectors along horizontal and vertical be denoted by **i** and **j**, then force P is $P\mathbf{i}$ and Q is $Q \cos \theta \mathbf{i}$ along the direction of **i** and $Q \sin \theta \mathbf{j}$ along **j**. The resultant is $P\mathbf{j}$.

$$\therefore \quad P\mathbf{j}=P\mathbf{i}+[Q \cos \theta\, \mathbf{i}+Q \sin \theta\, \mathbf{j}].$$

Equating the coefficients of **i** and **j**, we get

$$Q \sin \theta=P, \quad \text{and} \quad P+Q \cos \theta=0.$$

or $\quad Q \cos \theta=-P \quad \therefore \quad \tan \theta=-1 \quad$ or $\quad \theta=135°$

$$Q=\frac{P}{\sin \theta}=\frac{P}{\sin 135°}=\sqrt{2}P.$$

12. *Find the horizontal force and the force inclined at an angle of 60° to the vertical whose resultant is a vertical force of P lbs.-wt.*

Ans. $\sqrt{3}P$, $2P$.

13. *If two concurrent forces be represented by* $n.\overrightarrow{OP}$ *and* $m.\overrightarrow{OQ}$ *respectively, prove that their resultant is given by* $(m+n).\overrightarrow{OR}$ *where R divides PQ such that* $n.PR=m.RQ$.

$$\overrightarrow{OP}=\overrightarrow{OR}+\overrightarrow{RP};$$

$$\therefore \quad n.\overrightarrow{OP}=n.\overrightarrow{OR}+n.\overrightarrow{RP},$$

$$\overrightarrow{OQ}=\overrightarrow{OR}+\overrightarrow{RQ}.$$

$$\therefore \quad m.\overrightarrow{OQ}=m.\overrightarrow{OR}+m.\overrightarrow{RQ}.$$

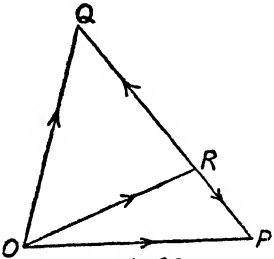


Fig. No. 20

$\therefore$ resultant of $n.\overrightarrow{OP}$ and $m.\overrightarrow{OQ}$ is

$$n.\overrightarrow{OP}+m.\overrightarrow{OQ}=(m+n)\,\overrightarrow{OR}+(n.\overrightarrow{RP}+m.\overrightarrow{RQ}).$$

Now we are given that $n.PR=m.RQ$.

$$\therefore \quad n.\overrightarrow{PR}=m.\overrightarrow{RQ} \quad \text{or} \quad -n.\overrightarrow{RP}=m.\overrightarrow{RQ}$$

or

$$n.\overrightarrow{RP}+m.\overrightarrow{RQ}=0.$$

$\therefore$ resultant is $(m+n)\,\overrightarrow{OR}$. The point R divides PQ in the ratio $m:n$.

Cor. In case the forces be $1.\overrightarrow{OP}$ and $1.\overrightarrow{OQ}$, then their

resultant will be (1+1) $\vec{OR}$ *i.e.* $2\vec{OR}$ where R divides PQ in the ratio 1 : 1 *i. e.* R is the middle point of PQ.

14. *Prove that the system of concurrent forces acting at a point and represented by* $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ *is equivalent to the system of forces represented by* $\vec{OD}$, $\vec{OE}$, $\vec{OF}$ *acting at the same point where* D, E, F *are the middle points of the sides* BC, CA *and* AB *respectively of the triangle* ABC.

We have to prove that $\vec{OA}+\vec{OB}+\vec{OC}=\vec{OD}+\vec{OE}+\vec{OF}$.

Now $\vec{OB}+\vec{OC}=2\vec{OD}$ where D is middle point of BC (by Q. 13). Write similar other relations and add.

15. (a) *Two forces act at the corner* A *of a quadrilateral* $ABCD$ *represented by* $\vec{AB}$, $\vec{AD}$ *and two at* C *represented by* $\vec{CB}$ *and* $\vec{CD}$. *Prove that their resultant is represented by* $4\vec{EF}$ *where* E *and* F *are the middle points of* AC *and* BD *respectively.*

(Agra M. Sc. 58) [Use cor. Q. 13]

Procced as in Q. 14.

(b) *ABCD is a quadrilateral and P the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of* $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, $\vec{OD}$ *is equal to* $4\vec{OP}$ *where* O *is any point.*

16. *Forces* P, Q *act at* O *and have a resultant* R. *If any transversal cuts their lines of action at* A, B *and* C *respectively, Prove that* $\dfrac{P}{OA}+\dfrac{Q}{OB}=\dfrac{R}{OC}$. **(Agra 36, 40, 45; Luck. B. Sc. 49)**

Let the forces P and Q be represented by $\overrightarrow{OL}$ and $\overrightarrow{OM}$ so that the diagonal $\overrightarrow{ON}$ represents the force R, so that

$$P+Q=R \quad \ldots \quad \ldots(1)$$

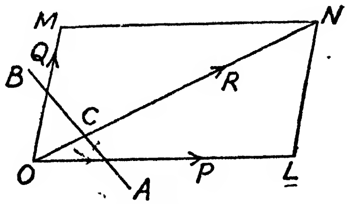


Fig. No. 21

Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$; $\therefore$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\mathbf{b}-\mathbf{a}$.

$$\therefore \quad \overrightarrow{AC}=k\overrightarrow{AB}=k\,.\,(\mathbf{b}-\mathbf{a}).$$

$$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\mathbf{a}+k\,(\mathbf{b}-\mathbf{a})=(1-k)\,\mathbf{a}+k\mathbf{b} \quad \ldots \quad \ldots(2)$$

P is in the same direction as $\mathbf{a}$ and hence it can be expressed as $m\,.\,\overrightarrow{OA}$. = P

Similarly $Q=n\,.\,\overrightarrow{OB}$ and let $R=t\,.\,\overrightarrow{OC}$.

$$\therefore \quad m=\frac{P}{OA},\ n=\frac{Q}{OB} \text{ and } t=\frac{R}{OC}.$$

We have to prove that $\dfrac{P}{OA}+\dfrac{Q}{OB}=\dfrac{R}{OC}$ or $m+n=t$.

From (1), we get $\quad m\,.\,\overrightarrow{OA}+n\,.\,\overrightarrow{OB}=t\,.\,\overrightarrow{OC} \quad \ldots \quad \ldots(3)$

or $$\frac{m\mathbf{a}}{t}+\frac{n\mathbf{b}}{t}=(1-k)\,\mathbf{a}+k\mathbf{b} \qquad \text{[from (2)]}.$$

Comparing $\mathbf{a}$ and $\mathbf{b}$, we get $\dfrac{m}{t}=1-k$ and $\dfrac{n}{t}=k$.

$$\therefore \quad \frac{m+n}{t}=1 \quad \text{or} \quad m+n=t. \qquad \text{Hence proved.}$$

Note. Alternative method for this question will be gevin in next chapter. **(Ex. 16 P. 70)**

Ex. 17. *Prove that the magnitude of the resultant of forces P_1 and P_2 is $(P_1^2+P_2^2+2P_1P_2\cos\theta)^{1/2}$, where θ is the angle between the direction of the forces.*

If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of the lines of action of the forces with reference to unit vectors **i**, **j** and **k**, then these forces can be expressed as

$$P_1l_1\mathbf{i}+P_1m_1\mathbf{j}+P_1n_1\mathbf{k} \text{ and } P_2l_2\mathbf{i}+P_2m_2\mathbf{j}+P_2n_2\mathbf{k}.$$

If R be their resultant, then R is given by

$$R=(P_1l_1+P_2l_2)\,\mathbf{i}+(P_1m_1+P_2m_2)\,\mathbf{j}+(P_1n_1+P_2n_2)\,\mathbf{k}.$$

The magnitude of R is

$$\sqrt{\{(P_1l_1+P_2l_2)^2+(P_1m_1+P_2m_2)^2+(P_1n_1+P_2n_2)^2\}}$$
$$=\{P_1^2(l_1^2+m_1^2+n_1^2)+P_2^2(l_2^2+m_2^2+n_2^2)+2P_1P_2(l_1l_2+m_1m_2+n_1n_2)\}^{1/2}$$
$$=\{P_1^2+P_2^2+2P_1P_2\cos\theta\}^{1/2}.$$

Note. If there be number of forces $P_1, P_2, P_3, \ldots$ then the magnitude is given by $\{\Sigma P_1^2+2\Sigma P_rP_s\cos(P_rP_s)\}^{1/2}$.

Ex. 18. *If O is the circum-centre and O′ the ortho-centre of a triangle ABC, then prove that*

(i) $\vec{OA}+\vec{OB}+\vec{OC}=\vec{OO'}$,

(ii) $\vec{O'A}+\vec{O'B}+\vec{O'C}=2\vec{O'O}$,

(ioi) $\vec{AO'}+\vec{O'B}+\vec{O'C}=\vec{AP}$,

where AP is the diameter of the circum-circle.

Before doing the above problem we should note that in the adjoining figure

$2OD=AO'$ (by geometry).

(i) $\vec{OB}+\vec{OC}=2\vec{OD}$, where D is the middle point of BC (Q. 13 Cor.).

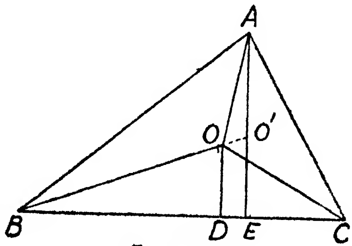


Fig. No. 22

$\therefore \quad \vec{OA}+\vec{OB}+\vec{OC}=\vec{OA}+2\vec{OD}=\vec{OA}+\vec{AO'}=\vec{OO'}.$

(ii) $\vec{O'B}+\vec{O'C}=2\vec{O'D}=2\,(\vec{O'O}+\vec{OD})=2\vec{O'O}+2\vec{OD}$

$$=2O'O+\vec{AO'}=2O'O-\vec{O'A},$$

$\therefore \quad \vec{O'A}+\vec{O'B}+\vec{O'C}=2O'O.$

(iii) $\vec{AO'}+\vec{O'B}+\vec{O'C}=2\vec{AO'}+(\vec{O'A}+\vec{O'B}+\vec{O'C})$ **(Note)**

$=2\vec{AO'}+2\vec{O'O}$ from (ii)$=2\,(\vec{AO'}+\vec{O'O})$

$=2AO=2.$ (the vector represented by the radius through A of the circum-circle)

$=\vec{AP}$ where AP is diameter.

19. *ABC is a triangle and P any point in BC. If PQ is the resultant of* $\vec{AP}, \vec{PB}, \vec{PC}$, *show that ABQC is a parallelogram and Q therefore a fixed point.* **(Luck. B. Sc. 54)**

$\vec{AP}+\vec{PB}+\vec{PC}$

$=\vec{AB}+\vec{PC}.$

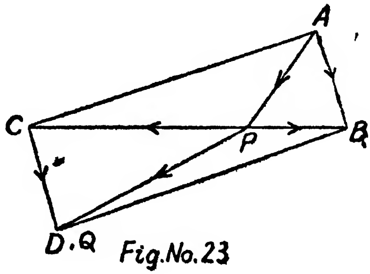


Fig.No.23

Now through C draw CD parallel and equal to AB. Therefore $ACDB$ is a parallelogram, so that

$\vec{AB}+\vec{PC}=\vec{PC}+\vec{CD}=\vec{PD}.$

But the resultant is given to be $\vec{PQ}$. Hence D coincides with Q. Since with the change of position of P in BC, AB

is not affected or *CD i. e. CQ* is not affected, therefore *Q* is fixed.

20. *A man travelling east at 8 miles per hour finds that the wind seems to blow directly from the north. On doubling his speed he finds that it appears to come from N. E. Find the velocity of the wind.*

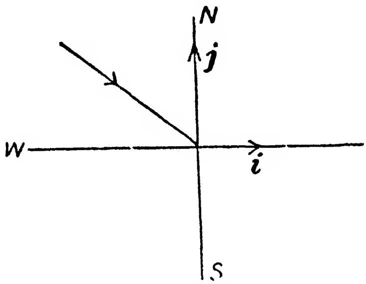


Fig. No 24

Let us suppose that unit vectors $\mathbf{i}$, $\mathbf{j}$ represent velocities of 8 m. p. h. towards *E.* and *N.* respectively so that the velocity of the man is represented by $\mathbf{i}$ as he is travelling at 8 m. p. h. towards east. Again suppose that the velocity of the wind in the $\mathbf{i}$ and $\mathbf{j}$ plane is given by $x\mathbf{i}+y\mathbf{j}$ where x and y are scalars whose values we are required to find.

Now the velocity of wind relative to the man is given by

actual velocity of wind+velocity of man reversed.

$$\therefore \quad (x\mathbf{i}+y\mathbf{j})-\mathbf{i}=(x-1)\,\mathbf{i}+y\mathbf{j}.$$

We are given that the wind appears to blow **from north** and hence it is given by $-t_1\mathbf{j}$.

$$\therefore \quad (x-1)\,\mathbf{i}+y\mathbf{j}=-t_1\mathbf{j}\,; \quad \therefore \quad x-1=0 \quad \text{or} \quad x=1.$$

Again when the man doubles his speed *i. e.* it becomes $2\mathbf{i}$ then the relative velocity is $(x\mathbf{i}+y\mathbf{j})-2\mathbf{i}$ or $(x-2)\,\mathbf{i}+y\mathbf{j}$.

But in this case we are given that it appears to blow from N. E. and is therefore given by $-t_2\,(\mathbf{i}+\mathbf{j})$.

$$\therefore \quad (x-2)\,\mathbf{i}+y\mathbf{j}=-t_2\,(\mathbf{i}+\mathbf{j}); \quad \therefore \quad x-2=-t_2$$

and $$y=-t_2.$$

Putting $x=1$ we find that $t_2=1$, and hence $y=-1$.

$\therefore$ velocity of wind is given by $\mathbf{i}-\mathbf{j}=-(\mathbf{j}-\mathbf{i})$. Its module is $\sqrt{2}$ units *i. e.* $8\sqrt{2}$ m. p. h. as one unit represents 8 m. p. h. and clearly the direction is from N. W.

21. *Show that if two vectors are parallel, the components of the one are proportional to those of the other. Hence or otherwise show that the three points (1, −2, −8), B (5, 0, −2), C (11, 3, 7) are collinear and find the ratio in which B divides AC.*

Ref. P. 13. Any of the two like vectors can be expressed as a multiple of the other. **Ans.** 2 : 3

22. *The vertices of a quadrilateral are A (1, 2, −1), B (−4, 2, −2), C (4, 1, −5) and D (2, −1, 3). At the point A, forces of magnitudes 2, 3, 2 lbs-wt. act along AB, AC, AD resrectively. Find their resultant.*

Refer Ex. 7 P. 23. Express in terms of unit vectors and then find unit vectors along AB, AC and AD ; then the force of 2 lbs. alog $AB=2.$unit vector along AB etc.

$$R=\frac{1}{\sqrt{26}}\mathbf{i}-\frac{9}{\sqrt{26}}\mathbf{j}-\frac{6}{\sqrt{26}}\mathbf{k} \text{ i. e. magnitude is } \sqrt{\left(\frac{118}{26}\right)}.$$

$\therefore$ direction cosines of result are

$$\frac{1}{\sqrt{(118)}}, \frac{-9}{\sqrt{(118)}}, \frac{-6}{\sqrt{(118)}}.$$

CHAPTER II

CENTROID, LINE AND PLANE

§ 1. *To find the position vector of the point P with divides the join of two given points A and B whose position vectors are* **a** *and b in a given ratio say m : n.*

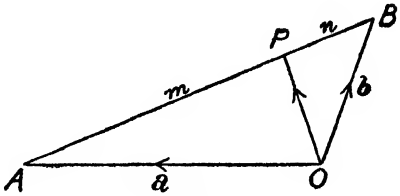


Fig. No. 25

We are given that

$$\frac{AP}{PB}=\frac{m}{n} \qquad \ldots(1)$$

Also $\vec{AB}=\vec{OB}-\vec{OA}$

$=\mathbf{b}-\mathbf{a}.$

Now $AP=\dfrac{m}{m+n}AB.$

$$\therefore \quad \vec{AP}=\frac{m}{m+n}\vec{AB}=\frac{m}{m+n}(\mathbf{b}-\mathbf{a})\,;$$

$$\therefore \quad \vec{OP}=\vec{OA}+\vec{AP}=\mathbf{a}+\frac{m}{m+n}(\mathbf{b}-\mathbf{a});$$

or $$\vec{OP}=\frac{n\mathbf{a}+m\mathbf{b}}{m+n}. \qquad \ldots \quad \ldots \quad \ldots(1)$$

Alternative. From (1), we get $n.\vec{AP}=m.\vec{PB}$

or $$n\,(\vec{OP}-\vec{OA})=m\,(\vec{OB}-\vec{OP})$$

or $$(m+n)\,\vec{OP}=n.\vec{OA}+m.\vec{OB}$$

$$\therefore \quad \vec{OP}=\frac{n\mathbf{a}+m\mathbf{b}}{m+n}.$$

Middle point. Putting $m=n=1$, we get the position vector of the middle point of AB as $\dfrac{\mathbf{a}+\mathbf{b}}{2}$ ... ...(3)

Note. The point P is called the **centroid** of the points A and B with **associated numbers n and m** respectively and divides AB internally in the ratio $m : n$,

Cartesian equivalence.

Let us suppose that in terms of unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ the point A is given by

$x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ so that the point A is (x_1, y_1, z_1)

and that of B is

$x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$ so that the point B is (x_2, y_2, z_2).

If P be the point $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then from (2), we get

$$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\frac{n(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})+m(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})}{m+n}$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, we get

$$\mathbf{x}=\frac{\mathbf{m}\mathbf{x}_2+\mathbf{n}\mathbf{x}_1}{\mathbf{m}+\mathbf{n}},\ \mathbf{y}=\frac{\mathbf{m}\mathbf{y}_2+\mathbf{n}\mathbf{y}_1}{\mathbf{m}+\mathbf{n}},\ \mathbf{z}=\frac{\mathbf{m}\mathbf{z}_2+\mathbf{n}\mathbf{z}_1}{\mathbf{m}+\mathbf{n}}.$$

§ 2. Centroid and centroid with associated numbers. (Agra M. Sc. 53, 58)

If there be n points whose position vectors relative to any origin O be given by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots$ then the point G whose position vector is

$$\overrightarrow{OG}=\frac{(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\ldots)}{n}$$

is called the centroid or centre of mean position of the given points.

Again if there be n scalars $p, q, r, s \ldots$ then the point G whose position vector is

$$\overrightarrow{OG}=\frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+s\mathbf{d}+\ldots}{p+q+r+s+\ldots}$$

is called the centroid of the given points with associated numbers $p, q, r, s\ldots$

Note :—The words **associated numbers** are used to cover up all forms of centroid, *e. g.* they may stand for the masses $m_1, m_2, m_3\ldots$ of a system of particles placed at given set of points; then the centroid is called the centre of mass, *i.e.* **c.m.** Similarly centre of gravity or centre of parallel forces. Thus if we place particles of masses $m_1, m_2, m_3\ldots$ at points $A, B, C, D\ldots$ whose position vectors are **a**, **b**, **c**, **d**..., then the position vector of their centre of mass G will be

$$\mathbf{r}=\overrightarrow{OG}=\frac{m_1\mathbf{a}+m_2\mathbf{b}+m_3\mathbf{c}+m_4\mathbf{d}+\ldots}{m_1+m_2+m_3+m_4+\ldots}\ldots \qquad \ldots(1)$$

Equivalence of Cartesian Forms.

Let us express the vectors **a**, **b**, **c**, **d** etc. in terms of unit vectors **i**, **j** and **k**, so that

$a=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ *i.e.* the point A is (x_1, y_1, z_1),
$b=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$ *i.e.* the point B is (x_2, y_2, z_2),
.............................and so on,

and let us suppose that the point $G=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, so that the coordinates of the centre of mass G are (x, y, z); then from (1), we get

$$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\frac{(\Sigma\, m_1x_1)\,\mathbf{i}+(\Sigma\, m_1y_1)\,\mathbf{j}+(\Sigma\, m_1z_1)\,\mathbf{k}}{\Sigma\, m_1}.$$

Equating the coefficients of **i**, **j** and **k**, we get the coordinates of the centre of mass as

$$\mathbf{x}=\frac{\Sigma\,\mathbf{m}_1\mathbf{x}_1}{\Sigma\,\mathbf{m}_1},\ \mathbf{y}=\frac{\Sigma\,\mathbf{m}_1\mathbf{y}_1}{\Sigma\,\mathbf{m}_1},\ \mathbf{z}=\frac{\Sigma\,\mathbf{m}_1\mathbf{z}_1}{\Sigma\,\mathbf{m}_1}.$$

Ex. 1. *Particles of masses 1, 2, 3, 4, 5, 6, 7, 8 grams respectively are placed at the corners of a unit cube, the first four at the corners A, B, C, D of one face and the last four at their*

projections A', B', C', D' respectively on the opposite face. Find. the coordinates of their centre of mass.

Take A as origin and let $\overrightarrow{AA'}$, $\overrightarrow{AB}$ and $\overrightarrow{AD}$ represent unit vectors **i**, **j** and **k** respectively.

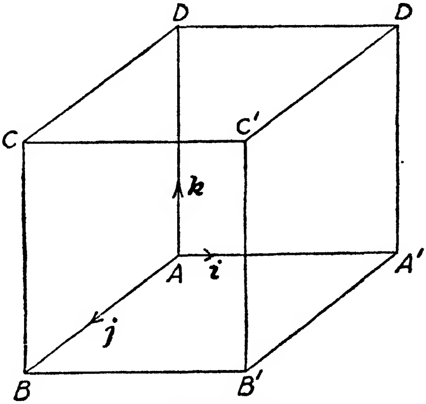


Fig. No. 26

The position vectors of the 8 corners are (the masses of the particles are also written within brackets–

(1) $A=0$ (5) $A'=\mathbf{i}$

(2) $B=\mathbf{j}$ (6) $B'=\mathbf{i}+\mathbf{j}$

(3) $C=\mathbf{j}+\mathbf{k}$ (7) $C'=\mathbf{i}+\mathbf{j}+\mathbf{k}$

(4) $D=\mathbf{k}$ (8) $D'=\mathbf{k}+\mathbf{i}$.

$\therefore$ position vector of the centre of mass say G is

$$\overrightarrow{AG}=\frac{1\cdot 0+2\cdot \mathbf{j}+3(\mathbf{j}+\mathbf{k})+4\mathbf{k}+5\mathbf{i}+6(\mathbf{i}+\mathbf{j})+7(\mathbf{i}+\mathbf{j}+\mathbf{k})+8(\mathbf{k}+\mathbf{i})}{1+2+3+4+5+6+7+8}$$

$$=\frac{26\mathbf{i}+18\mathbf{j}+22\mathbf{k}}{36}=\tfrac{1}{18}(13\mathbf{i}+9\mathbf{j}+11\mathbf{k}).$$

$\therefore$ magnitude of $AG=\frac{1}{18}\sqrt{(13^2+9^2+11^2)}=\frac{1}{18}\sqrt{(371)}$.

Also cooadinates of c.m. are $(\frac{13}{18}, \frac{9}{18}, \frac{11}{18})$.

§ **3.** *If G be the centroid of a system of points $A, B, C, \ldots$ whose position vectors are* **a, b, c...** *with associated numbers $p, q, r, \ldots$ and G' that of another system of points $A', B', C' \ldots$ whose position vectors are* $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *with associated numbers $p', q', r' \ldots$, then the centroid of all the points taken together is the*

centroid of the points G and G′ with associated numbers $(p+q+r\ldots)$ and $(p'+q'+r'+\ldots)$ respectively.

By definition, we have

$$\overrightarrow{OG}=\frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\ldots}{p+q+r+\ldots}$$

$$\overrightarrow{OG'}=\frac{p'\mathbf{a}'+q'\mathbf{b}'+r'\mathbf{c}'+\ldots}{p'+q'+r'+\ldots}.$$

Now if H be the centroid of the points G and G' with associated numbers $(p+q+r+\ldots)$ and $(p'+q'+r'+\ldots.)$ respectively, then

$$\overrightarrow{OH}=\frac{(p+q+r\ldots)\left(\dfrac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\ldots}{p+q+r+\ldots}\right)+(p'+q'+r'+\ldots)\dfrac{(p'\mathbf{a}'+q'\mathbf{b}'+r'\mathbf{c}'+\ldots}{p'+q'+r'+\ldots}}{(p+q+r+\ldots)+(p'+q'+r'+\ldots)}$$

or $$\overrightarrow{OH}=\frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+p'\mathbf{a}'+q'\mathbf{b}'+r'\mathbf{c}'+\ldots}{p+q+r+\ldots+p'+q'+r'+\ldots}.$$

H is clearly the centroid of all the points taken together.

§ **4.** *To prove that centroid is independent of the origin of vectors.* **(Agra M. Sc. 53, 58)**

Let G be the centroid of points whose position vectors relative to an origin O be **a**, **b**, **c**... with associated numbers $p, q, r\ldots.$ respectively.

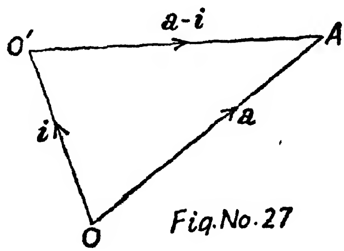


Fig. No. 27

$$\therefore \quad \overrightarrow{OG}=\frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\ldots}{p+q+r+\ldots}. \qquad \ldots \quad \ldots(1)$$

Now we have to prove that if instead of O we choose any point O' as origin, then the position of the centroid G will

remain unchanged. Let the position vector of O' be $\mathbf{i}$; then the position vector of A relative to O' will be

$$\overrightarrow{O'A}=\overrightarrow{OA}-\overrightarrow{OO'}=\mathbf{a}-\mathbf{i}.$$

Similarly position vectors of B, C, etc. relative to O' as origin are $\mathbf{b}-\mathbf{i}$, $\mathbf{c}-\mathbf{i}$...etc. If G' be the new position of C. G. relative to new origin O', then

$$O'G'=\frac{p\,(\mathbf{a}-\mathbf{i})+q\,(\mathbf{b}-\mathbf{i})+r\,(\mathbf{c}-\mathbf{i})+\dots}{p+q+r+\dots}$$

$$=\frac{(p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\dots)}{p+q+r+\dots}-\mathbf{i}$$

or $$\overrightarrow{O'G'}=\overrightarrow{OG}-\overrightarrow{OO'} \qquad \text{[from (1)]}$$

or $$O'G'=O'G.$$

Above relation shows that the point G' should coincide with G. Hence the position of the centroid remains unchanged with the change of origin.

§ 5. *To prove that the vector relation*

$$p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\dots=0$$

will be independent of the origin if and only if $p+q+r+\dots=0$, *where* p, q, r...*are scalars.*

With reference to the origin O, we have the relation

$$p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\dots=0\dots \qquad \dots \qquad \dots(1)$$

Now choose O' as origin whose position vector relative to O be $\mathbf{i}$; then as in last article the relation (1) w. r. t. new origin O' is

$$p\,(\mathbf{a}-\mathbf{i})+q\,(\mathbf{b}-\mathbf{i})+r\,(\mathbf{c}-\mathbf{i})+\dots \qquad \dots \qquad \dots(2)$$

or $$(p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\dots)-\mathbf{i}\,(p+q+r+\dots)\dots \qquad \dots(3)$$

If the relation (1) is to be independent of the origin then (1) and (3) should represent the same relation which is possible only if

$$p+q+r+\dots=0 \quad \dots \qquad \dots \qquad \dots(4)$$

Also if $p+q+r+\ldots=0$, then (1) and (3) represent the same. Hence proved.

Note. We have already proved that centroid is independent of the origin and we shall see that the above property is satisfied. Thus if **R** be the centroid of **a**, **b**, **c**... with associated numbers $p, q, r, \ldots$, then

$$\mathbf{R}=\frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\ldots}{p+q+r+\ldots}.$$

or $$p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\ldots-\mathbf{R}\,(p+q+r+\ldots)=0.$$

In the above relation the sum of the coefficients of the vectors is $(p+q+r+\ldots)-(p+q+r+\ldots)$ which is zero.

Exercise No. 2

1. *G is the centroid of tetrahedron ABCD. A'B'C'D' is another tetrahedron such that AA', BB', CC' and DD' are all bisected at G. Prove that G is also the centroid of the second tetrahedron.*

With respect to any origin O say the point G is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$ where **a**, **b**, **c**, **d** are the position vectors of A, B, C and D respectively. Let the position vectors of A', B', C', D' be $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, $\mathbf{d}'$ and we have to prove that

$$\overrightarrow{OG'}=\frac{\mathbf{a}'+\mathbf{b}'+\mathbf{c}'+\mathbf{d}'}{4}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}=\overrightarrow{OG}.$$

Now G is the mid. point of AA', BB', CC', DD'.

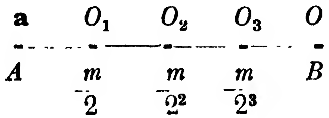
$$\therefore\quad \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}=\frac{\mathbf{a}+\mathbf{a}'}{2}=\frac{\mathbf{b}+\mathbf{b}'}{2}=\frac{\mathbf{c}+\mathbf{c}'}{2}=\frac{\mathbf{d}+\mathbf{d}'}{2}.$$

$$\therefore\quad \left.\begin{aligned}\mathbf{a}'&=\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}-\mathbf{a}}{2},\quad \mathbf{b}'=\frac{\mathbf{c}+\mathbf{d}+\mathbf{a}-\mathbf{b}}{2}\\ \mathbf{c}'&=\frac{\mathbf{d}+\mathbf{a}+\mathbf{b}-\mathbf{c}}{2},\quad \mathbf{d}'=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{d}}{2}\end{aligned}\right\}\ldots\qquad\ldots(A)$$

$$OG'=\frac{\mathbf{a}'+\mathbf{b}'+\mathbf{c}'+\mathbf{d}'}{4}=\tfrac{1}{4}.\frac{2\mathbf{a}+2\mathbf{b}+2\mathbf{c}+2\mathbf{d}}{2} \text{ from (A)}$$

$$=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}=\vec{OG}.$$

2. *A line AB is bisected in O_1, O_1B in O_2, O_2B in O_3 and so on ad infinitum. Particles of masses m, $\frac{1}{2}m$, $\frac{1}{2^2}m$...... are placed at O_1, O_2, O_3 ... etc. Show that the distance of their c. m. from B is equal to one-third of the distance from A to B.*



Let us choose B as origin, the point A be taken as $\mathbf{a}$ so that the points $O_1, O_2, O_3 \ldots$ are

$$\frac{\mathbf{a}}{2},\ \frac{1}{2}\left(\frac{\mathbf{a}}{2}\right)=\frac{\mathbf{a}}{2^2},\ \frac{1}{2}\left(\frac{\mathbf{a}}{2^2}\right)=\frac{\mathbf{a}}{2^3} \text{ etc.}$$

If G be the c. m. of the particles at these points, then

$$\vec{BG}=\frac{\frac{m}{2}\left(\frac{\mathbf{a}}{2}\right)+\frac{m}{2^2}\left(\frac{\mathbf{a}}{2^2}\right)+\frac{m}{2^3}\left(\frac{\mathbf{a}}{2^3}\right)+\ldots}{\frac{m}{2}+\frac{m}{2^2}+\frac{m}{2^3}+\ldots}$$

or

$$\vec{BG}=\frac{\frac{\mathbf{a}}{2^2}\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots\infty\right)}{\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\infty\right)} \quad \left\{\begin{array}{l}\text{Sum of G. P.}\\ =\dfrac{a}{1-r}\end{array}\right.$$

$$=\frac{\mathbf{a}}{2}\cdot\frac{4}{3}\cdot\frac{1}{2}=\frac{\mathbf{a}}{3}=\frac{1}{3}\vec{BA}.$$

Above shows that the distance of G from B is one-third that of A from B.

3. *Find the centroid of 3n points* $\mathbf{i}, 2\mathbf{i}, 3\mathbf{i} \ldots n\mathbf{i}$; $\mathbf{j}, 2\mathbf{j}, 3\mathbf{j}, \ldots n\mathbf{j}$; $\mathbf{k}, 2\mathbf{k}, 3\mathbf{k}, \ldots n\mathbf{k}$.

$$\frac{n+1}{6}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$ **Ans.**

4. *Particles of equal mass are placed at $(n-2)$ of the corners of a regular polygon of n sides. Find their c. m.*

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ ... be the position vectors of the vertices of the polygon with respect to centre O of the polygon. If equal masses were placed **at all the corners,** then the c. m. will coincide with O and hence

$$\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f}+\ldots}{n}=0.$$

Now let the vertices $\mathbf{a}$, $\mathbf{b}$ be vacant and G be the c. g. of the remaining $(n-2)$ particles placed at the other vertices so that

$$\overleftrightarrow{OG}=\frac{\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f}+\ldots}{n-2}=\frac{1}{n-2}\cdot\{-(\mathbf{a}+\mathbf{b})\} \text{ from (1)}$$

or $$\overrightarrow{OG}=\frac{2}{n-2}\left\{-\frac{\mathbf{a}+\mathbf{b}}{2}\right\}=-\frac{2}{n-2}\cdot\overrightarrow{OP}$$

where P is the mid. point of the join of vacant vertices,

or $$\overrightarrow{OG}=\frac{2}{n-2}\overrightarrow{PO} \quad \text{or} \quad \overrightarrow{PO}:\overrightarrow{OG}=n-2:2.$$

5. *A particle is acted on by a number of centres of forces, some of which attract and some repel, the force in each case varying as the distance. The intensities for different centres are different. Prove that the resultant passes through a fixed point for all positions of the particle.* **(Agra M. Sc. 40)**

Let O be the position of the particle and the various centres of force O_1, O_2, O_3 ... be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ ... w. r. t. O as origin. The forces acting on the particle are given by $\mu_1\mathbf{a}$, $\mu_2\mathbf{b}$, $\mu_3\mathbf{c}$ ... where μ_1, μ_2, μ_3 ... are constants which may be +ive

or $-$ive according as the centres attract or repel. The resultant of these forces is given by

$$\mu_1\mathbf{a}+\mu_2\mathbf{b}+\mu_3\mathbf{c}+\ldots \quad \ldots \quad \ldots \quad \ldots (1)$$

Now if G be the centroid of **a**, **b**, **c**... with associated numbers $\mu_1, \mu_2, \mu_3\ldots$, then

$$\frac{\mu_1\mathbf{a}+\mu_2\mathbf{b}+\mu_3\mathbf{c}+\ldots}{\mu_1+\mu_2+\mu_3\ldots}=\overrightarrow{OG}.$$

$$\therefore \quad \text{resultant}=(\mu_1+\mu_2+\mu_3+\ldots).\overrightarrow{OG}.$$

But G is independent of the origin of vectors O *i. e.* the particle. Hence G is fixed. Thus the resultant passes through a fixed point for all positions of the particle.

6. *If G_1 is the mean centre of A_1, B_1, C_1 and G_1 that of A_2, B_2, C_2, then show that*

$$\overrightarrow{A_1A_2}+\overrightarrow{B_1B_2}+\overrightarrow{C_1C_2}=3\overrightarrow{G_1G_2}.$$

Hint. If O be the origin, then

$$\overrightarrow{A_1A_2}=\overrightarrow{OA_2}-\overrightarrow{OA_1}=\mathbf{a}_2-\mathbf{a}_1.$$

7. *If* **i** *be the in-centre of the triangle ABC and a, b, c be the lengths of the sides, then prove that the forces*

$$a.\overrightarrow{\mathbf{iA}},\ b.\overrightarrow{\mathbf{iB}},\ c.\overrightarrow{\mathbf{iC}} \text{ are in equilibrium.}$$

We have to prove that

$$a.\overrightarrow{\mathbf{iA}}+b.\overrightarrow{\mathbf{iB}}+c.\overrightarrow{\mathbf{iC}}=0.$$

Now we know that the incentre of a triangle is the centroid of points A, B, C, with associated numbers a, b and c respectively. Also we know that centroid is independent of the origin of vectors. Therefore if we take **i** as origin of vectors, then the position vectors of the vertices are $\overrightarrow{\mathbf{i}A}$, $\overrightarrow{\mathbf{i}B}$ and $\overrightarrow{\mathbf{i}C}$ respectively and the position vectors of incentre **i** itself w. r. t. **i** as origin will be **0**.

$$\therefore \quad 0=\frac{a.\mathbf{i}\overrightarrow{A}+b.\mathbf{i}\overrightarrow{B}+c.\mathbf{i}\overrightarrow{C}}{a+b+c}$$

or
$$a.\mathbf{i}\overrightarrow{A}+b\ \mathbf{i}\overrightarrow{B}+c.\mathbf{i}\overrightarrow{C}=0.$$

Hence the forces are in equilibrium.

8. *The points D, E, F divide the sides BC, CA, AB of a triangle in the ratio 1 : 4, 3 : 2 and 3 : 7 respectively. Show that the sum of the vectors* $\overrightarrow{AD}, \overrightarrow{BE}, \overrightarrow{CF}$ *is parallel to* $\overrightarrow{CK}$ *where K divides AB in the ratio 1 : 3.*

9. *The vertices of a triangle ABC are A (2, −1, −3), B (4, 2, 3), C (6, 3, 4). Find the coordinates of points P and Q which divide BC in the ratio ±3 : 2 Also show that vector* $\overrightarrow{AQ}$ *has direction cosines proportional to 8, 6, 9.*

In terms of unit vectors **i, j** and **k** the position vector of A is $2\mathbf{i}-\mathbf{j}-3\mathbf{k}$ etc. § 7 P. 20. **Ans.** $P\left(\frac{26}{5}, \frac{13}{5}, \frac{18}{5}\right)$, Q (10, 5, 6).

10. *The position vectors of the vertices of a quadrilateral ABCD are* **a**, **b**, **c** *and* **d** *respectively. The diagonals AC and BD intersect at the point P which divides AC and BD in the ratios m : n and m′ : n′ respectively. Find the ratios in which the point Q which is the intersection of AB and CD divides these sides.*

The position vector of point P written in two different forms w. r. t. AC and BD are respectively

$$\frac{n\mathbf{a}+m\mathbf{c}}{m+n}=\frac{n'\mathbf{b}+m'\mathbf{d}}{m'+n'};$$

$$\therefore \quad n(m'+n')\,\mathbf{a}+m(m'+n')\,\mathbf{c}=n'(m+n)\,\mathbf{b}+m'(m+n)\,\mathbf{d}$$

or
$$n(m'+n')\,\mathbf{a}-n'(m+n)\,\mathbf{b}=m'(m+n)\,\mathbf{d}-m(m'+n')\,\mathbf{c}\ldots\text{(A)}$$

Now
$$n(m'+n')-n'(m+n)=nm'-n'm$$

and
$$m'(m+n)-m(m'+n')=nm'-n'm.$$

Hence dividing both sides of (A) by $nm'-n'm$, we get

$$\frac{n(m'+n')\,\mathbf{a}-n'(m+n)\,\mathbf{b}}{nm'-n'm}=\frac{m'(m+n)\,\mathbf{d}-m(m'+n')\,\mathbf{c}}{nm'-n'm}$$

L. H.S. represents a point on AB dividing it in the ratio $-n'(m+n) : n(m'+n')$ and R. H. S. represents a point on DC dividing it in the ratio $-m(m'+n') : m'(m+n)$.

Since the position vectors of both these points on AB and CD are same, ∴ hence it is their point of intersection *i.e.* Q.

§ 6. Vector equation of a straight line.

To find the vector equation of a straight line that passes through a given point **a** *and is parallel to a given vector* **b**.

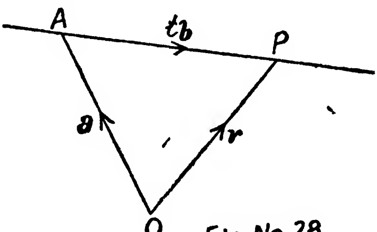


Fig.No.28

Let P be any point on the straight line through **a** which is parallel to **b** and its position vector be **r** so that $\mathbf{r}=\overrightarrow{OP}$. Now $\overrightarrow{AP}$ is parallel to **b** and hence it must be some multiple of **b**. Therefore $\overrightarrow{AP}=t\mathbf{b}$ where t is a constant positive for points on one side of A and negative for points on the other side.

Now $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$.

$$\therefore \quad \mathbf{r}=\mathbf{a}+t\mathbf{b} \qquad \ldots \quad \ldots \quad \ldots(1)$$

Every point P on the line is obtained for some value of t and for every value of t we get the position vector of a point on the line and hence above represents the vector equation of the required line.

Cor. 1. *To find the vector equation of a straight line through origin.*

Putting $\mathbf{a}=0$ in (1), we get the required line as $r=t\mathbf{b}$..(2)

Cor. 2. *To find the vector equation of a line through two given points A and B whose position vectors are* **a** *and* **b** *respectively.*

(Agra M. Sc. 42, 45, 47, 52)

Now $\vec{AB}=\vec{OB}-\vec{OA}=\mathbf{b}-\mathbf{a}$; therefore the required line is parallel to $\mathbf{b}-\mathbf{a}$ and passes through A.

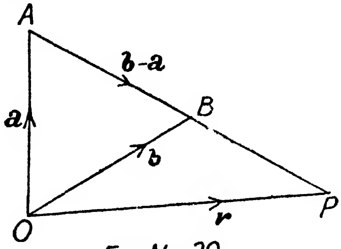


Fig. No. 29

Hence from (1) its equation is given by

$$r=\mathbf{a}+t\,(\mathbf{b}-\mathbf{a})\ldots(3)$$

or $$r=(1-t)\ \mathbf{a}+t\mathbf{b}.$$

Cor. 3. *To find the cartesian equivalents of the vector equation of the lines (1), (2), (3) found above.*

[Refer author's book on Solid Geometry]

Let in terms of unit vectors **i**, **j**, **k** through the origin O, the vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ so that P is the point (x, y, z) and $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ so that the point A is (a_1, a_2, a_3) and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ so that the point B is (b_1, b_2, b_3).

Substituting in (1), *i.e.* $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, we get.

$$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})+t\,(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}).$$

Equating the coefficients of **i**, **j** and **k**, we get

$$x=a_1+tb_1,\ y=a_2+tb_2,\ z=a_3+tb_3.$$

$$\therefore\quad \frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=t\ldots\quad\ldots\quad\ldots(4)$$

Above is the corresponding cartesian equation of a straight line through a given point (a_1, a_2, a_3) and whose direction cosines are proportional to b_1, b_2, b_3.

Again substituting in (2) *i.e.* $\mathbf{r}=t\mathbf{b}$ we get

$$\frac{x}{b_1}=\frac{y}{b_2}=\frac{z}{b_3}=t$$

which is the cartesian equation of a straight line through origin with direction cosines proportional to b_1, b_2, b_3.

Again substituting in (3), *i.e.* $\mathbf{r}=\mathbf{a}+t\,(\mathbf{b}-\mathbf{a})$ and proceeding as above, we get

$$\frac{x-a_1}{b_1-a_1}=\frac{y-a_2}{b_2-a_2}=\frac{z-a_3}{b_3-a_3}=t$$

which is the cartesian equation of a straight line through the points whose coordinates are (a_1, a_2, a_3) and (b_1, b_2, b_3).

§ 7. **Condition for three points to be collinear.**

(Agra M. Sc. 32, 37, 51)

To prove that the necessary and sufficient condition for three points in three dimensional space to be collinear is that there exists a linear relation connecting their position vectors such that the algebraic sum of the coefficients in it is zero.

We have seen that the equation of a straight line through $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{r}=\mathbf{a}+t\,(\mathbf{b}-\mathbf{a})=(1-t)\,\mathbf{a}+t\mathbf{b}$$

which may be written as

$$(1-t)\,\mathbf{a}+t\mathbf{b}-\mathbf{r}=0.$$

Above is a relation between the position vectors of three points A, B and P which are collinear and we observe that algebraic sum of the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ is $1-t+t-1$ which is zero. Hence the condition is necessary.

In order to prove that the condition is sufficient let us suppose that any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be connected by the relation $l\mathbf{a}+m\mathbf{b}+n\mathbf{c}=0$, where $l+m+n=0$ Dividing by n $(n\neq 0)$, we get

$$\frac{l}{n}\mathbf{a}+\frac{m}{n}\mathbf{b}+\mathbf{c}=0,\text{ where }\frac{l}{n}+\frac{m}{n}+1=0. \qquad \textbf{(Agra 45)}$$

Putting $\dfrac{m}{n}=-t$, we get $-\dfrac{l}{n}=1+\dfrac{m}{n}=1-t$.

$$\therefore\quad \mathbf{c}=-\frac{l}{n}\mathbf{a}-\frac{m}{n}\mathbf{b}\quad\text{or}\quad \mathbf{c}=(1-t)\,\mathbf{a}+t\mathbf{b}.$$

Above relation shows that $\mathbf{c}$ is a point on the line joining $\mathbf{a}$ and $\mathbf{b}$ and hence $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are collinear.

Alternative Proof.

Let us suppose that the points **a** **b**, **c** are collinear. Let **c** divide the line joining **a** and **b** in ratio $m : l$ so that

$$\mathbf{c}=\frac{l\mathbf{a}+m\mathbf{b}}{l+m}; \quad \therefore \quad l\mathbf{a}+m\mathbf{b}-(l+m)\,\mathbf{c}=0$$

or $l\mathbf{a}+m\mathbf{b}+n\mathbf{c}=0$, where $-(l+m)=n$ or $l+m+n=0$.

Again in order to prove that the condition is sufficient let us suppose that $l\mathbf{a}+m\mathbf{b}+n\mathbf{c}=0$, where $l+m+n=0$

or $$l\mathbf{a}+m\mathbf{b}=-n\mathbf{c}=(l+m)\,\mathbf{c}, \quad \because \quad l+m+n=0$$

or $$\frac{l\mathbf{a}+m\mathbf{b}}{l+m}=\mathbf{c}.$$

Above relation shows that **c** divides the join of **a** and **b** in the ratio $m : l$. Hence the three points **a**, **b**, **c** are collinear.

§ 8. Bisectors of the angle between two straight lines.

Let the equations of the lines AL and AM be given by

$$\mathbf{r}=\mathbf{a}+t\mathbf{b} \text{ and } \mathbf{r}=\mathbf{a}+t'\mathbf{c}$$

so that their point of intersection is the point whose position vector is **a** and they are parallel respectively to vectors **b** and **c**.

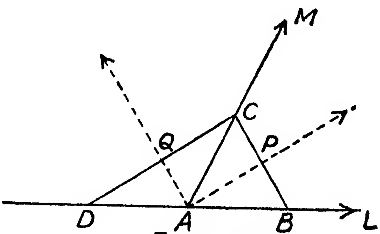


Fig No 30

Let the modules be denoted by corresponding italic letters a, b, c.

Now choose three points B, C and D on AL, AM and LA (produced) at a unit distance from A. If P and Q are the middle points of BC and CD, then AP and AQ are the required bisectors whose equations we are to find.

Now position vector of B is $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\mathbf{a}+\dfrac{\mathbf{b}}{b}$.

$\therefore$ $\overrightarrow{AB}$ is a unit vector along $\mathbf{b}$ and as such it is $\dfrac{\mathbf{b}}{b}$.

Similarly that of C is $\mathbf{a}+\dfrac{\mathbf{c}}{c}$ and that of D is $\mathbf{a}-\dfrac{\mathbf{b}}{b}$.

Therefore P and Q being the mid. points of BC and CD have their position vectors $\mathbf{a}+\frac{1}{2}\left(\dfrac{\mathbf{b}}{b}+\dfrac{\mathbf{c}}{c}\right)$, $\mathbf{a}+\frac{1}{2}\left(\dfrac{\mathbf{c}}{c}-\dfrac{\mathbf{b}}{b}\right)$ and A is $\mathbf{a}$. Now using the formula $\mathbf{r}=\mathbf{a}+t\,(\mathbf{b}-\mathbf{a})$ as the equation of a straight line through the points whose position vectors are $\mathbf{a}$ and $\mathbf{b}$, we get the equation of the lines AP and AQ as $\mathbf{r}=\mathbf{a}+\mathbf{k}.\frac{1}{2}\left(\dfrac{\mathbf{b}}{b}+\dfrac{\mathbf{c}}{c}\right)$

and $$\mathbf{r}=\mathbf{a}+\mathbf{k}'.\tfrac{1}{2}\left(\frac{\mathbf{c}}{c}-\frac{\mathbf{b}}{b}\right) \text{ respectively}$$

or $$\mathbf{r}=\mathbf{a}+t\left(\frac{\mathbf{b}}{b}+\frac{\mathbf{c}}{c}\right) \text{ and } \mathbf{r}=\mathbf{a}+t'\left(\frac{\mathbf{c}}{c}-\frac{\mathbf{b}}{b}\right).$$

Note. In case the point of intersection of the two lines were origin, then the corresponding equations would be

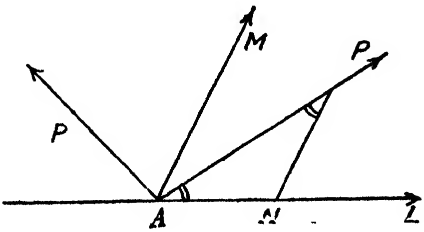
$$\mathbf{r}=t\left(\frac{\mathbf{b}}{b}+\frac{\mathbf{c}}{c}\right) \text{ and } \mathbf{r}=t'\left(\frac{\mathbf{c}}{c}-\frac{\mathbf{b}}{b}\right).$$

If $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are the unit vectors, then $\mathbf{b}=b\hat{\mathbf{b}}$ and $c=c\hat{\mathbf{c}}$.

$$\mathbf{r}=t\,(\hat{\mathbf{b}}+\hat{\mathbf{c}}) \text{ and } \mathbf{r}=t'\,(\hat{\mathbf{c}}-\hat{\mathbf{b}}).$$

Alternative method.

Taking the point A as origin the equations of the lines AL and AM are $\mathbf{r}=p\mathbf{b}$ and $\mathbf{r}=p'\mathbf{c}$. Take any pt. P on the internal bisector and draw PN parallel to the direction of AM. Therefore $\angle APN=\angle PAM$



$=\angle PAN$ and hence $AN=NP$. Now AN and NP are in the directions of vectors **b** and **c** and therefore they are the same scalar multiples of the corresponding unit vector in that direction,

i.e. if $\quad \vec{AN}=t\hat{\mathbf{b}}=t\,\dfrac{\mathbf{b}}{b}$, then $\vec{NP}=t\hat{\mathbf{c}}=t\,\dfrac{\mathbf{c}}{c}$.

$$\therefore\quad \mathbf{r}=\vec{AP}=\vec{AN}+\vec{NP}=\mathbf{t}\,(\hat{\mathbf{b}}+\hat{\mathbf{c}})=t\left(\frac{\mathbf{b}}{b}+\frac{\mathbf{c}}{c}\right).$$

Similarly the external bisector AP' is obtained by considering the internal bisectors of lines which are parallel to $-\mathbf{b}$ and $\mathbf{c}$ and therefore its equation is

$$\mathbf{r}=t\,(\hat{\mathbf{c}}-\hat{\mathbf{b}})=t\left(\frac{\mathbf{c}}{c}-\frac{\mathbf{b}}{b}\right).$$

Ex. 1. *Prove that the medians of a triangle are concurrent, and find the point of concurrency.* **(Lucknow 52, Agra 52, 55)**

1st Method.

Let **a, b, c** be the position vectors of the vertices of the triangle so that the co-ordinates of the mid. points D, E and F of the sides are

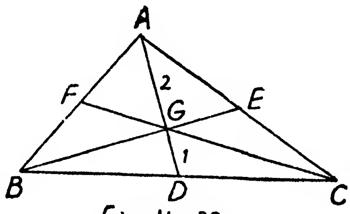


Fig. No. 32

$$\frac{\mathbf{b}+\mathbf{c}}{2},\ \frac{\mathbf{c}+\mathbf{a}}{2}\quad\text{and}\quad\frac{\mathbf{a}+\mathbf{b}}{2},$$

respectively.

Now A is $\mathbf{a}$ and D is $\dfrac{\mathbf{b}+\mathbf{c}}{2}$ and therefore the point G which divides AD in the ratio $2:1$ (*i.e.* the point of trisection) is

$$\frac{2.\dfrac{\mathbf{b}+\mathbf{c}}{2}+1.\mathbf{a}}{2+1}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}.$$

The symmetry of the result shows that the point G lies on the medians BE and CF as well and divides them in the ratio 2 : 1. Therefore the three medians are concurrent at the point $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$ which is also the centroid of the given triangle.

2nd Method.

We know that the vector equation of a line joining the points $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{r}=\mathbf{a}+t\,(\mathbf{b}-\mathbf{a})$

or $$\mathbf{r}=(1-t)\,\mathbf{a}+t\mathbf{b}.$$

A is $\mathbf{b}$, D is $\dfrac{\mathbf{b}+\mathbf{c}}{2}$, B is $\mathbf{b}$ and E is $\dfrac{\mathbf{c}+\mathbf{a}}{2}$.

Equations to medians AD and BE are

$$r=(1-t)\,\mathbf{a}+t\left(\frac{\mathbf{b}+\mathbf{c}}{2}\right) \quad \ldots \quad \ldots \quad \ldots(1)$$

and $$r=(1-s)\,\mathbf{b}+s\left(\frac{\mathbf{c}+\mathbf{a}}{2}\right) \quad \ldots \quad \ldots \quad \ldots(2)$$

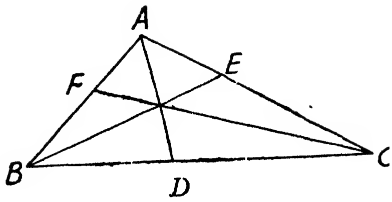
If the two straight lines intersect, we should be able to find some suitable values of s and t which should give identical values of $\mathbf{r}$. For this we shall compare the coefficients of equal vectors in the two values of $\mathbf{r}$.

$$\therefore \quad 1-t=\frac{s}{2},\ \frac{t}{2}=1-s,\ \frac{t}{2}=\frac{s}{2}.$$

Solving we get $t=s=\frac{2}{3}$. Substituting the values of t and s in (1) or (2) we observe that the medians AD and BE intersect at the point $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

Ex. 2. *Prove that the internal bisectors of the angles of a triangle are concurrent.*

(Agra 47, 52, 57; Lucknow 53; Dacca 27)



Let **a**, **b**, **c** be the position vectors of the vertices A, B and C respectively and α, β, γ be the lengths of the sides BC, CA and AB respectively. Also $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$ and $\overrightarrow{CA}=\mathbf{a}-\mathbf{c}$ and $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$.

Now by § 8, the equation of bisector AD is

$$\mathbf{r}=\mathbf{a}+p\left(\frac{\overrightarrow{AB}}{\gamma}+\frac{\overrightarrow{AC}}{\beta}\right)$$

$$\mathbf{r}=\mathbf{a}+p\left(\frac{\mathbf{b}-\mathbf{a}}{\gamma}+\frac{\mathbf{c}-\mathbf{a}}{\beta}\right)$$

or $$\mathbf{r}=\left(1-\frac{p}{\gamma}-\frac{p}{\beta}\right)\mathbf{a}+\frac{p}{\gamma}\mathbf{b}+\frac{p}{\beta}\mathbf{c}\ldots\quad\ldots(1)$$

Similarly we can write down the equations of the bisectors BE and CF as

$$\mathbf{r}=\left(1-\frac{q}{\alpha}-\frac{q}{\gamma}\right)\mathbf{b}+\frac{q}{\alpha}\mathbf{c}-\frac{q}{\gamma}\mathbf{a}\ \ldots\quad\ldots\quad\ldots(2)$$

and $$\mathbf{r}=\left(1-\frac{s}{\beta}-\frac{s}{\alpha}\right)\mathbf{c}+\frac{s}{\beta}\mathbf{a}-\frac{s}{\alpha}\mathbf{b}\ \ldots\quad\ldots\quad\ldots(2)$$

If (1), (2) intersect we should be able to find some suitable values of p, q, s which make the values of **r** identical.

For this we will compare the coefficients of vectors **a**, **b** and **c** in (1) and (2).

$$\therefore\ \left(1-\frac{p}{\gamma}-\frac{p}{\beta}\right)=\frac{q}{\gamma},\ 1-\frac{q}{\alpha}-\frac{q}{\gamma}=\frac{p}{\gamma}\text{ and }\frac{p}{\beta}=\frac{q}{\alpha}.$$

$$\therefore\ 1-p\frac{\beta+\gamma}{\beta\gamma}=\frac{1}{\gamma}\cdot\frac{\alpha p}{\beta}.\quad\therefore\ 1=p\frac{\alpha+\beta+\gamma}{\beta\gamma}$$

or $$p=\frac{\beta\gamma}{\alpha+\beta+\gamma};\quad\therefore\ q=\frac{\alpha}{\beta}\cdot p=\frac{\alpha}{\beta}\cdot\frac{\beta\gamma}{\alpha+\beta+\gamma}=\frac{\gamma\alpha}{\alpha+\beta+\gamma}.$$

On putting the value of p in (1) or of q in (2), we get the position vector of the point of intersection of (1) and (2) as

$$\mathbf{r}=\frac{\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}}{\alpha+\beta+\gamma}. \qquad \ldots \quad \ldots(4)$$

The symmetry of the result shows that this is also the point of intersection of the other bisectors and hence the three internal bisectors meet at the point $\dfrac{\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}}{\alpha+\beta+\gamma}$ which is the centroid of the points **a**, **b**, **c** with associated numbers α, β, γ, *i.e.* the lengths of the opposite sides.

(See alternative method in Ex. 5).

(b) *The internal bisector of an angle of a triangle and the external bisectors of the other two are concurrent.*

Let C be the origin and the position vectors of A and B be respectively **a** and **b**. Again let the lengths of the sides BC, CA and AB be a, b, c respectively.

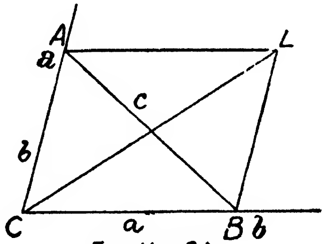


Fig. No. 34

Equation to the internal bisector of $\angle C$ is

$$\mathbf{r}=t\,(\hat{\mathbf{a}}+\hat{\mathbf{b}})=t\left(\frac{\mathbf{a}}{b}+\frac{\mathbf{b}}{a}\right) \qquad \ldots \quad \ldots(1)$$

Equation to the external bisector of $\angle A$, *i.e.* internal bisector of angle between $\overrightarrow{CA}$ produced and $\overrightarrow{AB}$ *i.e.* $\mathbf{b}-\mathbf{a}$ is by § 8,

$$\mathbf{r}=\mathbf{a}+s\left(\frac{\mathbf{a}}{b}+\frac{\mathbf{b}-\mathbf{a}}{c}\right)$$

Equation to external bisector of $\angle B$, *i.e.* internal bisector of angle between $\overrightarrow{CB}$ produced and $\overrightarrow{BA}$ *i.e.* $\mathbf{a}-\mathbf{b}$ is by § 8,

$$\mathbf{r}=\mathbf{b}+p\left(\frac{\mathbf{b}}{a}+\frac{\mathbf{a}-\mathbf{b}}{c}\right) \qquad \ldots \qquad \ldots(3)$$

Now (2) and (3) intersect at L say and hence on comparing, we have

$$1+s\left(\frac{1}{b}-\frac{1}{c}\right)=\frac{p}{c},$$

$$\frac{s}{c}=1+p\left(\frac{1}{a}-\frac{1}{c}\right)$$

or $$\frac{s}{c}=1+\left\{c+cs\left(\frac{c-b}{bc}\right)\right\}\left(\frac{c-a}{ac}\right).$$

$$\therefore \quad s\left(\frac{1}{c}-\frac{c^2-bc-ac+ab}{abc}\right)=1+\frac{c-a}{a}$$

or $$s\left(\frac{ab-c^2+bc+ac-ab}{abc}\right)=\frac{c}{a}.$$

or $$s(a+b-c)=bc$$

or $$s=\frac{bc}{a+b-c}.$$

Substituting the value of s in (2), we get the point L as

$$\left(1+\frac{s}{b}-\frac{s}{c}\right)\mathbf{a}+\frac{s}{c}\,\mathbf{b}$$

or $$\left(1+\frac{c}{a+b-c}-\frac{b}{a+b-c}\right)\mathbf{a}+\frac{b}{a+b-c}\,\mathbf{b}$$

or $$\frac{a}{a+b-c}\,\mathbf{a}+\frac{b}{a+b-c}\,\mathbf{b}.$$

This point will lie on (1), *i.e.*

$$\mathbf{r}=t\left(\frac{\mathbf{a}}{b}+\frac{\mathbf{b}}{a}\right)=t.\left(\frac{a\mathbf{a}+b\mathbf{b}}{ab}\right),$$

if we choose

$$t=\frac{ab}{a+b-c}.$$

Hence the three bisectors are concurrent.

Ex. 3. *Prove that the internal bisector of any angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.* **(Pb. 60, Lucknow B. Sc. 53)**

With reference to Ex. 2, the position vector of the point of intersection of the internal bisectors is

$$\frac{\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}}{\alpha+\beta+\gamma}=\frac{\alpha\mathbf{a}+(\beta+\gamma)\dfrac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}}{\alpha+\beta+\gamma} \quad \ldots \quad \ldots \quad \ldots (1)$$

Thus (i) is the centroid of the point $\mathbf{a}$ and $\dfrac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}$ with associated numbers α and $\beta+\gamma$ respectively. Now $\mathbf{a}$ corresponds to the point A; therefore $\dfrac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}$ is the position vector of the point D where the internal bisector of the angle A meets BC. But $\dfrac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}$ is the centroid of points B and C with associated numbers β and γ and therefore divides BC in the ratio $\gamma : \beta$ or $AB : AC$. Hence proved.

Alternative.

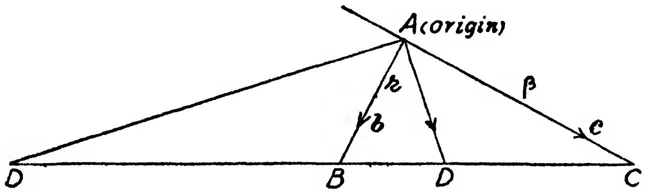


Fig. No. 35

Taking A as origin, let the position vectors of B and C be $\mathbf{b}$ and $\mathbf{c}$ respectively and let $AB=\gamma$ and $AC=\beta$.

The equation of the internal bisector AD is

$$\mathbf{r}=t\,(\hat{\mathbf{b}}+\hat{\mathbf{c}})=t\left(\frac{\mathbf{b}}{\gamma}+\frac{\mathbf{c}}{\beta}\right)$$

or $$\mathbf{r}=t.\frac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta\gamma} \qquad \ldots \quad \ldots \quad \ldots(1)$$

For all values of t, we get a point on the bisector. Choosing $t=\frac{\beta\gamma}{\beta+\gamma}$, we get the point $\frac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}$ on it, but it is the centroid of the points **b** and **c** with associated numbers β and γ and is therefore a point on BC.

Hence $t=\frac{\beta\gamma}{\beta+\gamma}$ gives the point D which clearly divides BC in the ratio $\gamma : \beta$ or $AB : AC$.

Note. 1. Writing the equation of BC as

$$\mathbf{r}=(1-p)\,\mathbf{b}+p\mathbf{c}$$

and solving with (1), we can find the position vector of point D direct as found above.

2. By considering the external bisector,

$$\mathbf{r}=t\,(\hat{\mathbf{b}}-\hat{\mathbf{c}})=t\left(\frac{\mathbf{b}}{\gamma}-\frac{\mathbf{c}}{\beta}\right)$$

and its intersection with BC is at the point $\frac{\beta\mathbf{b}-\gamma\mathbf{c}}{\beta-\gamma}$. We can now show that external bisector divides the opposite side externally in the ratio of the sides containing the angle.

Ex. 4. *ABC is a triangle in which the internal and external bisectors of angle A meet the opposite side BC in D and D'. A' is the middle point of DD'. Similarly B' and C' are the middle points of EE' and FF' respectively. Show that the points A', B' and C' are collinear.*

With reference to Ex. 2, the position vectors of D and D' are $\frac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}$ and $\frac{\beta\mathbf{b}-\gamma\mathbf{c}}{\beta-\gamma}$ respectively.

$\therefore\ A'$ is $\frac{1}{2}\left(\frac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}+\frac{\beta\mathbf{b}-\gamma\mathbf{c}}{\beta-\gamma}\right)=\frac{1}{2}\left(\frac{\beta^2\mathbf{b}-\gamma^2\mathbf{c}}{\beta^2-\gamma^2}\right)=\mathbf{a}'$ say.

Similarly the position vectors of B' and C' are

$$\tfrac{1}{2}\left(\frac{\gamma^2\mathbf{c}-\alpha^2\mathbf{a}}{\gamma^2-\alpha^2}\right)=\mathbf{b}' \text{ say and } \tfrac{1}{2}\left(\frac{\alpha^2\mathbf{a}-\beta^2\mathbf{b}}{\alpha^2-\beta^2}\right)=\mathbf{c}' \text{ say.}$$

Now we know by § 7 **P. 49** that the three points whose position vectors are $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ will be collinear if there exists a relation of the form $L\mathbf{a}'+M\mathbf{b}'+N\mathbf{c}'=0$ such that

$$L+M+N=0.$$

Now $\quad (\beta^2-\gamma^2)\,\mathbf{a}'+(\gamma^2-\alpha^2)\,\mathbf{b}'+(\alpha^2-\beta^2)\,\mathbf{c}'$

i. e. $L\mathbf{a}'+M\mathbf{b}'+N\mathbf{c}'=\tfrac{1}{2}\,(\beta^2\mathbf{b}-\gamma^2\mathbf{c}+\gamma^2\mathbf{c}-\alpha^2\mathbf{a}+\alpha^2\mathbf{a}-\beta^2\mathbf{b})=0$

and also $(\beta^2-\gamma^2)+(\gamma^2-\alpha^2)+(\alpha^2-\beta^2)=0$, *i. e.* $L+M+N=0$.

Hence the three points A', B' and C' are collinear.

Ex. 5. *Prove that internal bisectors of the angles of a triangle are concurrent.*

We have already proved that the internal bisector divides the opposite side in the ratio of the sides containing the angle. Thus if the internal bisector of A meets BC in D, then $D=\dfrac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}$ and A is $\mathbf{a}$.

Therefore the position vector of a point which divides AD in the ratio $\beta+\gamma : \alpha$ is

$$\frac{\alpha\mathbf{a}+(\beta+\gamma)\dfrac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}}{\alpha+\beta+\gamma} \quad \text{or} \quad \frac{\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}}{\alpha+\beta+\gamma}.$$

The symmetry of the result shows that it also lies on the other bisectors. Hence the internal bisectors are concurrent.

Ex. 6. (i) *The lines joining the vertices of tetrahedron to the centroids of area of opposite faces are concurrent.*

(Agra 43, 53; Rajputana 56)

(ii) *The joins of the mid. points of the opposite edges of a tetrahedron intersect and bisect each other.* **(Agra 34, Utkal 53)**

Let the position vectors of the points A, B, C, D be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively and G_1 be the centroid of B, C, D so that G_1 is $\dfrac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}$, and A is $\mathbf{a}$.

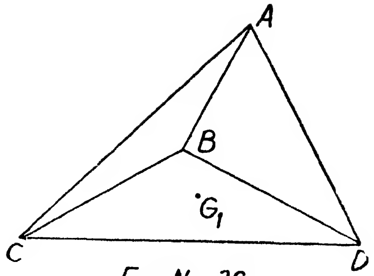


Fig. No. 36

Now the position vector of the point which divides AG_1 in the ratio $3:1$ is

$$\frac{3.\dfrac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}+1.\mathbf{a}}{3+1}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}.$$

The symmetry of the result shows that the point will be on BG_2, CG_3 and DG_4. Hence these lines concur at the point $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$ which is the centroid of the tetrahedron.

Alternative :—

The vector equations of the lines AG_1 and BG_2 are :—

$$\mathbf{r}=(1-t)\,\mathbf{a}+t\left(\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}\right) \quad \ldots \quad \ldots \quad \ldots(1)$$

$$\mathbf{r}=(1-s)\,\mathbf{b}+s\left(\frac{\mathbf{c}+\mathbf{d}+\mathbf{a}}{3}\right) \quad \ldots \quad \ldots \quad \ldots(2)$$

If they intersect, then for some suitable values of t and s the corresponding values of r should be identical. Comparing, we get $1-t=\dfrac{s}{3}$, $1-s=\dfrac{t}{3}$, $\dfrac{t}{3}=\dfrac{s}{3}$, $\dfrac{t}{3}=\dfrac{s}{3}$.

$$\therefore \quad t=s=\tfrac{3}{4}.$$

Putting in (1) or (2), we get the point of intersection as $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$ and the symmetry shows that it lies on other lines also.

(ii) The mid. point of DA is $\dfrac{\mathbf{a}+\mathbf{d}}{2}$ and that of BC is

$\dfrac{\mathbf{b}+\mathbf{c}}{2}$ and the mid. point of these mid. points is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$ and symmetry of the result proves the theorem. We can also prove the same by considering the vector equation.

Ex. 7. *The four diagonals of a parallelopiped and the joins of the mid. points of opposite edges are concurrent at a common point of bisection.* **(Agra 40, 55)**

Taking O as origin, let the position vectors of A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively so that those of D, E and F are $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, $\mathbf{c}+\mathbf{a}$ respectively and that of G is $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

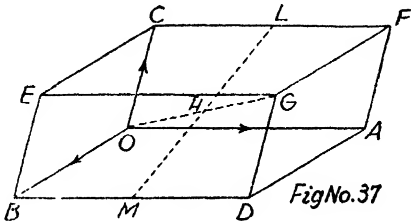


Fig No. 37

If M_1 be the mid. point of diagonal OG, then M_1 is

$$\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}.$$

If M_2 be the mid. point of diagonal AE and M_2 is

$$\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}, \quad i.e. \ M_1.$$

Similarly mid. point of other diagonals DC and BF is also the point whose position vector is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$.

Again mid. point L of BD is $\frac{1}{2}(\mathbf{b}+\mathbf{a}+\mathbf{b})=\dfrac{\mathbf{a}+2\mathbf{b}}{2}$

and mid. point M of CF is $\frac{1}{2}(\mathbf{c}+\mathbf{c}+\mathbf{a})=\dfrac{2\mathbf{c}+\mathbf{a}}{2}$.

$\therefore$ mid. point G of LM is $\frac{1}{2}\left[\dfrac{\mathbf{a}+2\mathbf{b}}{2}+\dfrac{2\mathbf{c}+\mathbf{a}}{2}\right]$

$$=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$$

which is same as the mid. point of diagonals.

In a similar manner we can show that the mid. point of the join of mid. points of opposite sides is also given by $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$.

Hence proved.

Ex. 8. *If M, N are the mid. points of the sides AB, CD of a parallelogram ABCD, prove that DM and BN cut the diagonal AC at its points of trisection which are also the points of trisection of DM and BN respectively.* **(Agra 48)**

Taking A as origin let the position vectors of B, C and D be $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively so that position vctors of M and N are $\frac{\mathbf{b}}{2}$ and $\frac{\mathbf{c}+\mathbf{d}}{2}$.

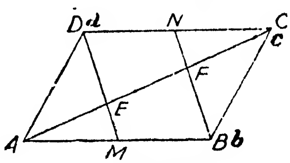


Fig No. 38

Now equation to AC is $\mathbf{r}=t\mathbf{c}=t(\mathbf{b}+\mathbf{d})$... ... ...(1)

$$\because \quad \vec{AC}=\vec{AB}+\vec{BC} \quad i.e. \quad \mathbf{c}=\mathbf{b}+\mathbf{d}.$$

Again equation of DM is $\mathbf{r}=(1-s)\,\mathbf{d}+s.\frac{\mathbf{b}}{2}$... ...(2)

Equation of NB is $\mathbf{r}=(1-p)\,\frac{\mathbf{c}+\mathbf{d}}{2}+p.\mathbf{b}$

or $$\mathbf{r}=(1-p)\,\frac{\mathbf{b}+2\mathbf{d}}{2}+p\mathbf{b} \quad \ldots \quad \ldots(3)$$

$$\because \quad \mathbf{c}=\mathbf{b}+\mathbf{d}.$$

(1) and (2) intersect at E, therefore we should be able to find suitable values of t and s which should give identical values of r; comparing the coefficients, we get

$$t=\frac{s}{2}=1-s. \quad \therefore \quad s=\tfrac{2}{3} \text{ and } t=\tfrac{1}{3}.$$

$$\therefore \quad \text{point } E \text{ is } \frac{\mathbf{b}+\mathbf{d}}{2}=\frac{\mathbf{c}}{3} \quad \text{or} \quad \vec{AE}=\tfrac{1}{3}\vec{AC}.$$

Again $\overrightarrow{DE}=\overrightarrow{AE}-\overrightarrow{AD}=\dfrac{\mathbf{b}+\mathbf{d}}{3}-\mathbf{d}=\dfrac{\mathbf{b}-2\mathbf{d}}{3}=\frac{2}{3}\left(\dfrac{\mathbf{b}}{2}-\mathbf{d}\right)$

or $\qquad \overrightarrow{DE}=\frac{2}{3}.(\overrightarrow{AM}-\overrightarrow{AD})=\frac{2}{3}\overrightarrow{DM}.$

Similarly we can prove that $\overrightarrow{AF}=\frac{2}{3}\,\overrightarrow{AC}$ and $\overrightarrow{BF}=\frac{2}{3}\,\overrightarrow{BN}$.

Ex. 9. *ABCD is a parallelogram. M and N are the mid-points of the sides AB and BC respectively. Prove that DM and DN trisect the diagonal AC.* **(Agra 48, Lucknow 51)**

Ex. 10. *Three concurrent straight lines OA, OB, OC are produced to D, E, F respectively. Show that the points of intersection of AB and DE, BC and EF, CA and FD are collinear.* **(Agra 45)**

Let us choose the point O as origin and the points A, B, C as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ so that the points D, E and F may be taken $t_1\mathbf{a}$, $t_2\mathbf{b}$, $t_3\mathbf{c}$ respectively.

Equation to AB is $\mathbf{r}=(1-p)\,\mathbf{a}+p\mathbf{b}$ ... ...(1)

Equation ty DE is $\mathbf{r}=(1-q)\,t_1\mathbf{a}+q\,.t_2\mathbf{b}$ ... ...(2)

If they intersect say at X_1, we should be able to find values of p and q which give us identical values for $\mathbf{r}$. Hence comparing,

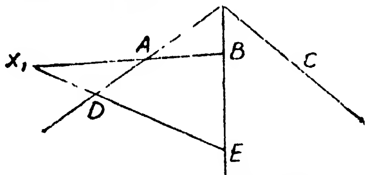


Fig. No. 39

$$1-p=(1-q)\,t_1 \text{ and } p=qt_2.$$

$$\therefore\quad 1-qt_2=t_1-qt_1 \quad\text{or}\quad q\,(t_1-t_2)=(t_1-1)$$

or $$q=\frac{t_1-1}{t_1-t_2};\quad \therefore\quad p=\frac{t_2\,(t_1-1)}{t_1-t_2}.$$

Substitutiug the value of p in (1) or q in (2), we get the position vector of the point X_1 which say is $\mathbf{r}_1$ *i. e.* $\overrightarrow{OX_1}$.

$$\therefore\quad \mathbf{r}_1=\frac{1-t_2}{t_1-t_2}\,t_1\mathbf{a}+\frac{(t_1-1)\,t_2\mathbf{b}}{t_1-t_2}$$

Donated by

or $$\frac{(t_1-t_2)}{(1-t_1)(1-t_2)}.\mathbf{r}_1=\frac{t_1}{1-t_1}\mathbf{a}-\frac{t_2}{1-t_2}\mathbf{b}. \quad \ldots \quad \ldots(1)$$

Similarly if r_2, r_3 be the position vectors of the points of the intersection of other pair of lines, then proceeding as above,

$$\frac{(t_2-t_3)}{(1-t_2)(1-t_3)}\mathbf{r}_2=\frac{t_2}{1-t_2}\mathbf{b}-\frac{t_3}{1-t_3}\mathbf{c},\ldots \quad \ldots \quad \ldots(4)$$

$$\frac{(t_3-t_1)}{(1-t_3)(1-t_1)}\mathbf{r}_3=\frac{t_3}{1-t_3}\mathbf{c}-\frac{t_1}{1-t_1}\mathbf{a}\ldots \quad \ldots \quad \ldots(5)$$

Now we know that three points whose position vectors are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ will be collinear if there exists a relation of the form $x\mathbf{r}_1+y\mathbf{r}_2+z\mathbf{r}_3=0$ where $x+y+z=0$. Adding (3), (4) and (5) (§ 7 P. 49),

$$\frac{t_1-t_2}{(1-t_1)(1-t_2)}\mathbf{r}_1+\frac{t_2-t_3}{(1-t_2)(1-t_3)}\mathbf{r}_2+\frac{t_3-t_1}{(1-t_3)(1-t_1)}\mathbf{r}_3=0$$

i. e. $x\mathbf{r}_1+y\mathbf{r}_2+z\mathbf{r}_3=0$ where $x+y+z=\Sigma\dfrac{t_1-t_2}{(1-t_1)(1-t_2)}$

which is clearly zero. Hence the points whose position vectors are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are collinear.

Ex. 11. *Prove the converse of the last exercise, i. e. if the points of intersection are real and collinear, then DA, EB and FC are concurrent.*

Let O be the point of intersection of DA and EB. Join O to C and we must prove that F also lies on OC. Taking O as origin, let us choose that A, B, C are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and D and E are $t_1\mathbf{a}$ and $t_2\mathbf{b}$. In case F lies on OC, then position vector. of F should come out to be $t_3\mathbf{c}$. Let the position vector of F be $\quad x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$.

BC is $\quad r=(1-p)\,\mathbf{b}+p\mathbf{c}$,

EF is $\quad r=(1-q)\,t_2\mathbf{b}+q\,(x\mathbf{a}+y\mathbf{b}+z\mathbf{c})$,

BC and EF intersect. On comparing the coefficients,

$$1-p=(1-q)\,t_2+qy \text{ and } p=qz \text{ and } qx=0.$$

When $qx=0$, we get either $q=0$ or $x=0$. But if $q=0$ then clearly p is also zero and we get from 1st that $t_2=1$ which

shows that B and E coincide which is impossible and hence $q \neq 0$ but $x=0$. Similarly considering the points of intersection of CA and DF, we get $y=0$. Therefore the position vector of F is given by $z\mathbf{c}=z\overrightarrow{OC}$. Hence F lies on OC or FC also passes through O.

12. *Using the vector equation of a straight line, show that the mid. points of the diagonals of a complete quadrilateral are collinear. Establish also the harmonic property of the figure i. e. each diagonal of a complete quadrilateral is cut harmonically by the other two.*

(a) Let $ABCD$ be any quadrilateral, two of its diagonals being AC and BD. Let BA and CD meet at E and AD and BC meet at F; then $ABCDEF$ is a complete quadrilateral and EF is its third diagonal. Let P, Q and R be the middle points of the diagonals BD, AC and EF respectively and we have to prove that P, Q, R are collinear.

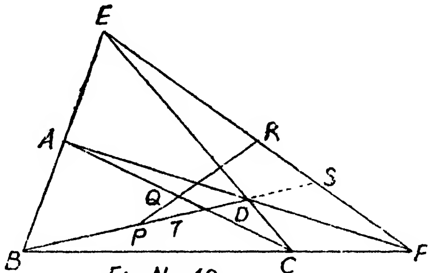


Fig. No. 40

Choose E as origin of coordinates and let the position vectors of A and C be $\mathbf{a}$ and $\mathbf{c}$ respectively, so that if B and D may be taken as $p\mathbf{a}$ and $q\mathbf{c}$ respectively, the equation to BC joining $p\mathbf{a}$ and $\mathbf{c}$ is

$$r=(1-t)\,.\,p\mathbf{a}+t\mathbf{c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

The equation to AD joining $\mathbf{a}$ and $q\mathbf{c}$ is

$$r=(1-s)\,\mathbf{a}+s\,.\,q\mathbf{c} \quad \ldots \quad \ldots \quad \ldots(2)$$

Now (1) and (2) intersect at F and hence on comparing

we have

$$(1-t)\,p=1-s \quad \ldots \quad \ldots \quad \ldots \quad \ldots(3)$$

$$t=sq \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots(4)$$

$$\therefore \quad (1-sq)\,p=1-s \quad \text{or} \quad s\,(1-pq)=1-p.$$

$$\therefore \quad s=\frac{1-p}{1-pq} \text{ and } t=\frac{(1-p)\,q}{1-pq}$$

Substituting the values of t in (1) or of s in (2), we get the position vector of the point F as

$$\frac{p\,(1-q)}{1-pq}\,\mathbf{a}+\frac{q\,(1-p)}{1-pq}\,\mathbf{c} \quad \ldots \quad \ldots \quad \ldots(5)$$

Now P, Q, R are the mid. points of AC, BD and EF respectively so that their position vectors are say $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$.

$$\left.\begin{aligned} \mathbf{r}_1&=\tfrac{1}{2}\,(\mathbf{a}+\mathbf{c}) \text{ and } \mathbf{r}_2=\tfrac{1}{2}\,(p\mathbf{a}+q\mathbf{c}) \\ \mathbf{r}_3&=\tfrac{1}{2}\left[\frac{p\,(1-q)}{1-pq}\,\mathbf{a}+\frac{q\,(1-p)}{1-pq}\,\mathbf{c}\right] \end{aligned}\right\} \ldots \quad \ldots(A)$$

Now if $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ be collinear then $x\mathbf{r}_1+y\mathbf{r}_2+z\mathbf{r}_3=0$ where $x+y+z=0$ and x, y, z are any three scalars.

Now $(1-pq)\,\mathbf{r}_3=\frac{1}{2}\,[(p-pq)\,\mathbf{a}+(q-pq)\,\mathbf{c}]$

$$=\tfrac{1}{2}\,(p\mathbf{a}+q\mathbf{c})-\frac{pq}{2}\,(\mathbf{a}+\mathbf{c})$$

or $\quad (1-pq)\,\mathbf{r}_3=\mathbf{r}_2-pq\mathbf{r}_1 \quad$ [from (A)]

or $\quad pq\mathbf{r}_1-\mathbf{r}_2+(1-pq)\,\mathbf{r}_3=0$

where $\quad pq-1+1-pq=0.$

Hence $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are collinear. (by § 7 **P. 49**)

(b) Now let the diagonal BD be cut by the diagonals AC and EF in points T and S respectively ; then the points B, T, D, S will form a harmonic pencil if BT, BD and BS are in H. P. or $\frac{1}{BT}$, $\frac{1}{BD}$ and $\frac{1}{BS}$ are in A. P. *i. e.* $\frac{1}{BT}+\frac{1}{BS}=\frac{2}{BD}$.

Now equation to AC joining $\mathbf{a}$ and $\mathbf{c}$ is

$$\mathbf{r}=(1-t)\,\mathbf{a}+t\mathbf{c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

Equation to BD joining $p\mathbf{a}$ and $q\mathbf{c}$ is

$$r=(1-s)p\mathbf{a}+s.q\mathbf{c} \qquad \ldots \quad \ldots \quad \ldots(2)$$

Equation to EF joining origin to point F given by (5) of part (a) is

$$r=\mathbf{k}\left[\frac{p(1-q)}{1-pq}\mathbf{a}+\frac{q(1-p)}{1-pq}\mathbf{c}\right] \qquad \ldots \quad \ldots \quad \ldots(3)$$

AC and BD intersect at T and hence on comparing (1) and (2), we get

$$1-t=(1-s)p \text{ and } t=qs.$$

$\therefore \quad (1-qs)=(1-s)p$ or $\dfrac{p-1}{p-q}=s$ and $1-s=\dfrac{1-q}{p-q}$.

Substituting in (2), we get the position vector of T as

$$\overrightarrow{ET}=\frac{(1-q)}{p-q}\mathbf{a}+\frac{(1-q)}{p-q}q\mathbf{c}.$$

$$\therefore \quad \overrightarrow{BT}=\overrightarrow{ET}-\overrightarrow{EB}=\left[\frac{(1-q)}{p-q}p\mathbf{a}-\frac{(1-p)}{p-q}q\mathbf{c}\right]-p\mathbf{a}$$

$$=\frac{1}{p-q}[p(1-q-p+q)\mathbf{a}-q(1-p)\mathbf{c}].$$

$$\overrightarrow{BT}=\frac{1-p}{p-q}[p\mathbf{a}-q\mathbf{c}].$$

$\therefore \quad$ length BT is the module of $\overrightarrow{BT}$.

$$\frac{1}{BT}=\frac{p-q}{1-p}\cdot\frac{1}{|p\mathbf{a}-q\mathbf{c}|} \qquad \ldots \quad \ldots \quad \ldots(4)$$

Again BD and EF intersect at S and hence on comparing (2) and (3), we get

$$(1-s)p=\frac{\mathbf{k}p(1-q)}{1-pq} \text{ and } sq=\frac{\mathbf{k}q(1-p)}{1-pq}$$

or $\quad 1-\dfrac{\mathbf{k}(1-p)}{1-pq}=\dfrac{\mathbf{k}.(1-q)}{1-pq} \quad$ or $\quad 1-pq=\mathbf{k}(1-p+1-q)$.

$$\therefore \quad \mathbf{k}=\frac{1-pq}{2-p-q}.$$

$\therefore$ position vector of S *i. e.* $\overrightarrow{ES}$

$$=\frac{1-pq}{2-p-q}\times\left[\frac{p(1-q)}{1-pq}\mathbf{a}+\frac{q(1-p)}{1-pq}\mathbf{c}\right].$$

$$\therefore\quad \overrightarrow{BS}=\overrightarrow{ES}-\overrightarrow{EB}=\frac{1-pq}{2-p-q}\left[\frac{p(1-q)}{1-pq}\mathbf{a}+\frac{q(1-p)}{1-pq}\mathbf{c}\right]-p\mathbf{a}$$

$$=\frac{1}{2-p-q}\left[p(1-q)\mathbf{a}+q(1-p)\mathbf{c}-p(2-p-q)\mathbf{a}\right]$$

$$=\frac{1}{2-p-q}\left[p(p-1)\mathbf{a}+q(1-p)\mathbf{c}\right]=\frac{p-1}{2-p-q}\left[p\mathbf{a}-q\mathbf{c}\right].$$

$$\therefore\quad \frac{1}{BS}=\frac{2-p-q}{p-1}\frac{1}{|p\mathbf{a}-q\mathbf{c}|}\qquad\ldots\qquad\ldots\qquad\ldots(5)$$

Again $\overrightarrow{BD}=\overrightarrow{ED}-\overrightarrow{EB}=q\mathbf{c}-p\mathbf{a}=-(p\mathbf{a}-q\mathbf{c})$.

$$\therefore\quad \frac{2}{BD}=-2\frac{1}{|p\mathbf{a}-q\mathbf{c}|}\qquad\ldots\qquad\ldots\qquad\ldots(6)$$

Now from (4) and (5), we get

$$\frac{1}{BT}+\frac{1}{BS}=\frac{1}{|p\mathbf{a}-q\mathbf{c}|}\left[\frac{p-q}{1-p}+\frac{2-p-q}{p-1}\right]$$

$$=\frac{1}{|p\mathbf{a}-q\mathbf{c}|}\left[\frac{p-q-2+p+q}{1-p}\right]$$

$$=-2\frac{1}{|p\mathbf{a}-p\mathbf{c}|}=\frac{2}{BD}\text{ from (6).}$$

Hence proved.

Theorem of Pappus

Ex. 13. (a) *If there be two sets of collinear points A_1, A_2, A_3; B_1, B_2, B_3; then prove that the points of intersection of the pair of lines A_1B_2, A_2B_1, A_2B_3, A_3B_2, A_3B_1, A_1B_3 are collinear.*

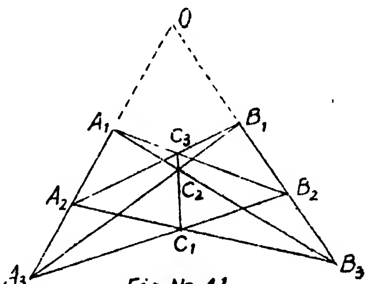


Fig No.41

Let O be the point of intersection of $A_3A_2A_1$ and $B_3B_2B_1$ which may be taken as origin. Again suppose that A_1, A_2, A_3 are $p_1\mathbf{a}$, $p_2\mathbf{a}$

$p_3\mathbf{a}$ and B_1, B_2, B_3 are $q_1\mathbf{b}$, $q_2\mathbf{b}$, $q_3\mathbf{b}$. If C_1, C_2, C_3 are points of intersection of the three pairs of lines, we have to prove that C_1, C_2, C_3 are collinear.

Equation to A_2B_3 joining $p_2\mathbf{a}$ and $q_3\mathbf{b}$ is

$$\mathbf{r}=(1-t)\,p_2\mathbf{a}+tq_3\mathbf{b} \qquad \ldots \quad \ldots(1)$$

Equation to A_3B_2 joining $p_3\mathbf{a}$ and $q_2\mathbf{b}$ is

$$\mathbf{r}=(1-s)\,p_3\mathbf{a}+sq_2\mathbf{b} \qquad \ldots \quad \ldots(2)$$

Since they intersect at C_1, we have on comparing

$$(1-t)\,p_2=(1-s)\,p_3 \text{ and } tq_3=sq_2; \quad \therefore\ s=t\,\frac{q_3}{q_2}.$$

$$\therefore\quad (1-t)\,p_2=\left(1-t\frac{q_3}{q_2}\right)p_3 \quad \text{or} \quad p_2-p_3=t\left(p_2-\frac{q_3p_3}{q_2}\right).$$

$$\therefore\quad t=\frac{q_2\,(p_2-p_3)}{p_2q_2-p_3q_3} \text{ and } 1-t=\frac{p_3\,(q_2-q_3)}{p_2q_2-p_3q_3}.$$

If $\mathbf{r}_1$ be the position vector of the point C_1, then on substituting for t and $(1-t)$ in (1), we get

$$\mathbf{r}_1\,(p_2q_2-p_3q_3)=p_2p_3\,(q_2-q_3)\,\mathbf{a}+q_2q_3\,(p_2-p_3)\,\mathbf{c}.$$

Multiply both sides by p_1q_1,

or $\mathbf{r}_1.p_1q_1\,(p_2q_2-p_3q_3)=p_1p_2p_3.q_1\,(q_2-q_3)\,\mathbf{a}+q_1q_2q_3.p_1\,(p_2-p_3)\,\mathbf{b}$.

Similarly if $\mathbf{r}_2$ and $\mathbf{r}_3$ be the position vectors of other points of intersection, we have

$$\mathbf{r}_2.p_2q_2\,(p_3q_3-p_1q_1)=p_1p_2p_3.q_2\,(q_3-q_1)\,\mathbf{a}+q_1q_2q_3.p_2\,(p_3-p_1)\,\mathbf{b},$$

$$\mathbf{r}_3.p_3q_3\,(p_1q_1-p_2q_2)=p_1p_2p_3.q_3\,(q_1-q_2)\,\mathbf{a}+q_1q_2q_3.p_3\,(p_1-p_2)\,\mathbf{b}.$$

$$\therefore\quad \Sigma\,\mathbf{r}_1.p_1q_1\,(p_2q_2-p_3q_3)=0,\ i.e.\ L\mathbf{r}_1+M\mathbf{r}_2+N\mathbf{r}_3=0$$

where $\Sigma\,p_1q_1\,(p_2q_2-p_3q_3)$ *i.e.* $L+M+N$ is also zero. Hence the three points whose position vectors are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are collinear.

Theorem of Desargues.

Ex. 13. (b) *If ABC, $A_1B_1C_1$ be two triangles such that the lines AA_1, BB_1, CC_1 are concurrent, then the points of intersection of pairs of sides BC, B_1C_1; CA, C_1A_1; AB, A_1B_1 are collinear.*

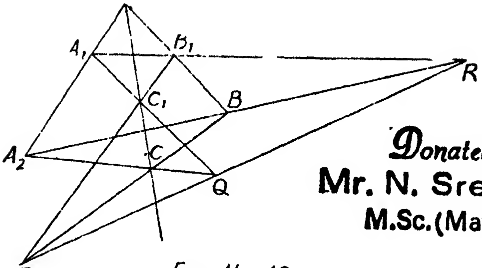


Fig No 42

Note :—For *Ceva and Menelaus Theorems* see Q. 9 P. 93, and Q. 10 P. 95.

Ex. 14. *Through the middle point P of the side AD of a parallelogram ABCD, the straight line BP is drawn cutting AC at R and CD produced at Q. Prove that QR=2RB.*

Take A as origin and B and D as **b** and **d** respectively. Now proceed exactly as in Ex. 12, 2nd part.

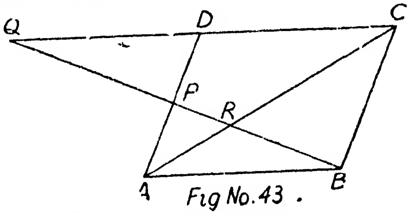


Fig No.43 .

Ex. 15. *The median AD of a triangle ABC is bisected at E and BE is produced to meet the side AC in F. Prove that*

$$AF=\tfrac{1}{3}AC \text{ and } EF=\tfrac{1}{4}BF.$$

Ex. 16. *Forces P, Q act at O and have a resultant R. If any transversal cuts their lines of action in A, B and C respectively, then prove that* $\dfrac{P}{OA}+\dfrac{Q}{OB}=\dfrac{R}{OC}$. **(Agra 49, Lucknow 42)**

Let the forces P and Q be represented by $\overrightarrow{OL}$ and $\overrightarrow{OM}$ so that the diagonal $\overrightarrow{ON}$ represents the force R so that

$$P+Q=R \qquad \ldots(1)$$

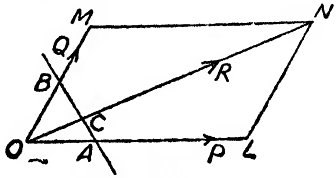


Fig No.44

Again suppose that $P = m.\vec{OA}$ and $Q = n.\vec{OB}$ and let $R = t.\vec{OC}$.

$$\therefore \quad m = \frac{P}{OA},\ n = \frac{Q}{OB} \text{ and } t = \frac{R}{OC} \quad \ldots \quad \ldots(2)$$

Now substituting for P, Q and R in (1), we get

$$m.\vec{OA} + n.\vec{OB} = t.\vec{OC} \quad \ldots \quad \ldots \quad \ldots(3)$$

or $$m.\vec{OA} + n.\vec{OB} - t.\vec{OC} = 0.$$

But A, B, C are collinear and as such sum of the coefficients of their position vectors in a relation connecting them should be zero.

$$\therefore \quad m + n - t = 0 \quad \text{or} \quad m + n = t.$$

Putting the values of m, n and t from (2), we get the required result.

Ex. 17. Prove that the straight lines joining the mid. points of two non-parallel sides of a trapezium is parallel to the parallel sides and half of their sum.

We have to prove that PQ is parallel to AB and equal to $\frac{1}{2}(AB + DC)$.

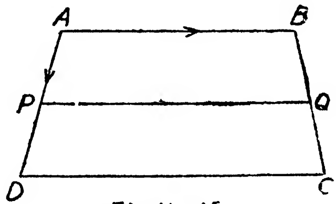


Fig. No. 45

Taking A as origin, let the position vectors of B and D be **b** and **d** respectively.

Now DC is $\parallel$ to AB; $\therefore \ \vec{DC} = t.\vec{AB} = t.\mathbf{b}$, where

$$\frac{DC}{AB} = t \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

$\therefore$ position vector of C is $\vec{AC} = \vec{AD} + \vec{DC} = \mathbf{d} + t\mathbf{b}$.

P being the mid. point AD has its position vector $\mathbf{d}/2$ and Q being the mid. point of BC has its position vector $\frac{1}{2}(\mathbf{b}+\mathbf{d}+t\mathbf{b})$.

$$\therefore \quad \overrightarrow{PQ}=\overrightarrow{AQ}-\overrightarrow{AP}=\tfrac{1}{2}(\mathbf{b}+\mathbf{d}+t\mathbf{b})-\frac{\mathbf{d}}{2}=\tfrac{1}{2}(1+t)\,\mathbf{b}$$

$$=\tfrac{1}{2}(1+t)\,\overrightarrow{AB}.$$

Since $\overrightarrow{PQ}$ is some scalar multiple of $\overrightarrow{AB}$, hence PQ is parallel to AB and hence to DC.

Also $\quad \dfrac{PQ}{AB}=\frac{1}{2}(1+t)=\frac{1}{2}\left(1+\dfrac{DC}{AB}\right)=\frac{1}{2}\dfrac{(AB+DC)}{AB}$ from (1).

$$\therefore \quad PQ=\tfrac{1}{2}(AB+DC).$$

Ex. 18. *Prove that the straight line joining the mid. points of the diagonals of a trapezium is parallel to parallel sides and half of their difference.*

Ex. 19. *Prove that in any triangle the line joining the mid. points of any two sides is parallel to the third side and half of its length.* **(Agra 56, Rajputana B. Sc. 60)**

Ex. 20. *Prove that the diagonals of a parallelogram bisect each other and, conversely, if the diagonals of a quadrilateral bisect each other, it is a parallelogram.* **(Agra 36, Lucknow B. Sc. 54)**

Ex. 21. *Show that the figure formed by joining the mid. points of the sides of a quadrilateral taken in order is a parallelogram.*

(Lucknow 48)

Ex. 22. *If a straight line is drawn parallel to the base of a triangle, the line joining the vertex to the intersection of the diagonals of the trapezium so formed bisects the base of the triangle.*

(Agra 59)

Taking A as origin let the position vectors of B and C be $\mathbf{b}$ and $\mathbf{c}$ respectively.

Now NM is parallel to BC.

$\therefore \quad \frac{AN}{AB}=\frac{AM}{AC}=x$ say.

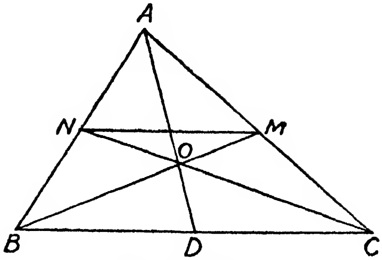


Fig. No. 46

Therefore position vectors of N and M are $x\mathbf{b}$ and $x\mathbf{c}$ respectively.

Equation to BM is $\mathbf{r}=(1-t)\,\mathbf{b}+t.x\mathbf{c}\ldots \quad \ldots(1)$

Equation to CN is

$$\mathbf{r}=(1-s)\,\mathbf{c}+s.\,x\,\mathbf{b}$$

Comparing $1-t=sx$ and $1-s=tx$

or $\quad 1-s=x\,(1-sx)$; $\therefore \quad s=\frac{1-x}{1-x^2}=\frac{1}{1+x}$.

$$\therefore \quad 1-s=\frac{x}{1+x}.$$

Hence the point O is given by $\frac{x}{1+x}\,(\mathbf{b}+\mathbf{c})$.

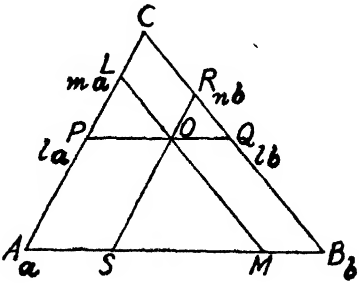
Equation to AO is $\mathbf{r}=\frac{x}{1+x}\,(\mathbf{b}+\mathbf{c})\ldots \quad \ldots. \quad .. \quad \ldots(3)$

Again equation to AD where D is mid. point of BC is

$$\mathbf{r}=\tfrac{1}{2}\,(\mathbf{b}+\mathbf{c}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots(4)$$

Relations (3) and (4) show that D lies on AO produced.

(b) *If through any point within a triangle, lines be drawn parallel to the sides, the sum of the ratios of these lines to the corresponding sides is 2.* **(Agra 51, 61)**



Take the vertex C as origin let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$ respectively and that of point O within the $\triangle$ be $\mathbf{c}$.

Since PQ is to $\parallel AB$ therefore if position vectors of P be $l\mathbf{a}$, then then that of

B will be $l\mathbf{b}$ and also $\dfrac{PQ}{AB}=l$... ... ...(1)

Again let the position vector of L on CA be $m\mathbf{a}$.

$\therefore \quad LA=(1-m)\,\mathbf{a}.$

Now in $\triangle ABC$, LM is drawn parallel to BC.

$\therefore \quad \dfrac{LM}{BC}=\dfrac{LA}{CA}=1-m$.. ... ...(2)

Similarly let the position vector of R on CB be $n\mathbf{b}$

$\therefore \quad RB=(1-n)\,\mathbf{b}.$

$\therefore \quad \dfrac{RS}{CA}=\dfrac{RB}{CB}=1-n$.. ... ...(3)

Now PQ passes through $l\mathbf{a}$ and $l\mathbf{b}$.

Its equation is $\mathbf{r}=(1-t)\,l\mathbf{a}+tl\mathbf{b}$... ... ...(4)

RS passes through R *i.e.* $n\mathbf{b}$ and is parallel to CA *i.e.* $\mathbf{a}$.

Its equation is $\mathbf{r}=n\mathbf{b}+s\mathbf{a}$ ... ...(5)

LM passes through L *i. e.* $m\mathbf{a}$ and is parallel to CB *i.e.* $\mathbf{b}$.

Its equation is $\mathbf{r}=m\mathbf{a}+u\mathbf{b}$... ... ...(6)

Since all the three lines (4), (5) and (6) intersect at O, therefore we have on comparing the coefficients,

$(1-t)\,l=s,\ tl=n$ [from (4) and (5)]

and $n=u,\ s=m$ [from (5) and (6)].

Now sum of the ratios from (1), (2) and (3) is

$$\begin{aligned} l+1-m+1-n &= 2+l-(m+n) \\ &= 2+l-(tl+s) \\ &= 2+(1-t)\,l-s=2+0=2. \end{aligned}$$

Hence proved.

Ex. 23. *Prove that the sides about the equal angles of equi-angular triangles are proportional.*

Let us consider two equiangular triangles ABC and ADE having a common vertex at A. We have to prove that

$$\frac{AB}{AD}=\frac{AC}{AE}=\frac{BC}{DE}.$$

Let $\qquad \vec{AB}=\mathbf{c},\ \vec{BC}=a,\ \vec{CA}=\mathbf{b}.$

$\therefore \quad \mathbf{a}+\mathbf{b}+\mathbf{c}=0 \ldots (1)$

$\therefore \quad \overrightarrow{AD}=k_3.\mathbf{c}.$

$\therefore \quad \dfrac{AD}{AB}=k_3, \ \overrightarrow{DE}=k_1 a.$

$\therefore \quad \dfrac{DE}{BC}=k_1, \ \overrightarrow{EA}=k_2.\mathbf{b}$

$\therefore \quad \dfrac{EA}{CA}=k_2.$

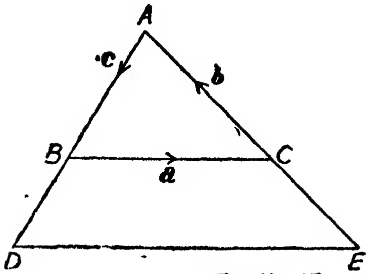


Fig. No 47

Again $\quad \overrightarrow{AD}+\overrightarrow{DE}+\overrightarrow{EA}=0.$

$\therefore \quad k_3\mathbf{c}+k_1\mathbf{a}+k_2\mathbf{b}=0$

or $\quad -k_3(\mathbf{a}+\mathbf{b})+k_1\mathbf{a}+k_2\mathbf{b}=0 \qquad$ [from (1)]

or $\quad (k_1-k_3)\,\mathbf{a}+(k_2-k_3)\,\mathbf{b}=0 \quad \ldots \quad \ldots \quad \ldots (2)$

Now we know from § 5 page 16 chapter I that if there exists a relation of the form $x\mathbf{a}+y\mathbf{b}=0$ between two non-collinear vectors, then $x=0$, $y=0$. Hence from (2), we get

$$k_1-k_3=0, \ k_2-k_3=0. \quad \therefore \quad k_1=k_2=k_3.$$

or $$\frac{AD}{AB}=\frac{DE}{BC}=\frac{EA}{CA}.$$ **Hence proved.**

Ex. 24. *In a parallelogram ABCD a point P is taken on the side AD, such that $n.AP=AD$. The line BP cuts the diagonal AC AC in the point Q. Prove that $(n+1)\,AQ=AC$.*

(Lucknow B. Sc. Supp. 48)

Taking A as origin let $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be the position vectors of B, C and D respectively and since $\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$, we have

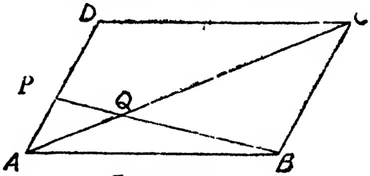


Fig. No. 48

$$\mathbf{b}+\mathbf{d}=\mathbf{c}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

$$\overrightarrow{AP}=\frac{1}{n}\overrightarrow{AD}=\frac{1}{n}.\mathbf{d},\ \overrightarrow{AQ}=x.\overrightarrow{AC} \text{ say.}$$

Hence from (1), we get $\overrightarrow{AB}+n\overrightarrow{AP}-\frac{1}{x}\overrightarrow{AQ}=0.$

Above is a relation between the position vectors of three collinear points; hence sum of the coefficients should be zero.

$$\therefore\ 1+n-\frac{1}{x}=0 \text{ or } x=\frac{1}{n+1}; \quad \therefore\ \overrightarrow{AQ}=\frac{1}{n+1}\overrightarrow{AC}$$

or $$(n+1)\,AQ=AC.$$

Ex. 25. *ABC is any triangle and O any point in the plane of the same. AB, BO, CO meet the sides BC, CA and AB in D, E, F respectively. Prove that*

$$\frac{OD}{AD}+\frac{OE}{BE}+\frac{OF}{CF}=1.$$

Ex. 26. **(a)** *Prove that the points*

$\mathbf{a}-2\mathbf{b}+3\mathbf{c},\ 2\mathbf{a}+3\mathbf{b}-4\mathbf{c},\ -7\mathbf{b}+10\mathbf{c}$ *are collinear.*

Let the above points be A, B, C respectively relative to any origin O. Now if we can find three scalars L, M, N such that $L.\overrightarrow{OA}+M.\overrightarrow{OB}+N.\overrightarrow{OC}=0$ where $L+M+N=0$, then A, B, C are collinear. Choosing $L=2$, $M=-1$, $N=-1$, we find that $2(\mathbf{a}-2\mathbf{b}+3\mathbf{c})-1(2\mathbf{a}+3\mathbf{b}-4\mathbf{c})-1(-7\mathbf{b}+10\mathbf{c})=0$, where $2-1-1=0$. Hence collinear.

Alternative.

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(-7\mathbf{b}+10\mathbf{c})-(\mathbf{a}-2\mathbf{b}+3\mathbf{c})$$
$$=-\mathbf{a}-5\mathbf{b}+7\mathbf{c}.$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(-7\mathbf{b}+10\mathbf{c})-(2\mathbf{a}+3\mathbf{b}-4\mathbf{c})$$
$$=2\mathbf{a}-10\mathbf{b}+14\mathbf{c}.$$

We observe that $\overrightarrow{BC}=2\overrightarrow{AC}$. $\therefore$ B, A, C are collinear.

(b) *Prove that the points whose position vectors are* $\mathbf{a}, \mathbf{b}$, $3\mathbf{a}-2\mathbf{b}$ *are collinear.* **(Delhi 50, Banaras 54, Agra 55, 58)**

Ex. 27. (a) *Find the vector equation of the line joining the points* $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ *and* $3\mathbf{k}-2\mathbf{j}$.

Ans. $\mathbf{r}=(\mathbf{i}-2\mathbf{j}+\mathbf{k})+t\,(\mathbf{i}-2\mathbf{k})$,

(b) *Show that line joining the points* $A\,(2, -3, -1)$ *and* $B\,(8, -1, 2)$ *has equations* $\dfrac{x-2}{6}=\dfrac{y+3}{2}=\dfrac{z+1}{3}$. *Find two points on the line whose distance from A is 14.*

Ans. $(14, 1, 5)$ and $(-10, -7, -7)$.

Ex. 28. *Prove that the line joining the points* $6\mathbf{a}-4\mathbf{b}+4\mathbf{c}$, $-4\mathbf{c}$ *and the line joining* $-\mathbf{a}-2\mathbf{b}-3\mathbf{c}$, $\mathbf{a}+2\mathbf{b}-5\mathbf{c}$ *intersect at* $-4\mathbf{c}$.

Ex. 29. *In the triangle ABC, points Q and R are taken in the sides CA and AB respectively such that* $CQ=QA$ *and* $AR=2RB$. *BQ and CZ intersect at O. Prove that* $CO=3OR$. *Also find the ratio in which AO divides BC.*

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of the points A, B and C respectively so that under given conditions the position vectors of Q and R are $\dfrac{\mathbf{c}+\mathbf{a}}{2}$ and $\dfrac{2\mathbf{b}+\mathbf{a}}{3}$.

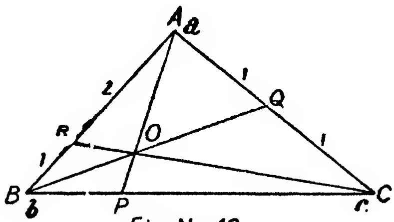


Fig. No. 49

Equations to BQ is $\mathbf{r}=(1-t)\,\mathbf{b}+t\,\dfrac{\mathbf{c}+\mathbf{a}}{2}$ ... ...(1)

Equation to CR is $\mathbf{r}=(1-s)\,\mathbf{c}+s\,\dfrac{2\mathbf{b}+\mathbf{a}}{3}$ ... ...(2)

Now (1) and (2) intersect at O and hence on comparing, we get

$$\frac{t}{2}=\frac{s}{3},\ 1-t=\frac{2s}{3},\ \frac{t}{2}=1-s \quad \therefore \quad s=\tfrac{3}{4} \text{ and } t=\tfrac{1}{2}.$$

The position vector of O is therefore $\dfrac{\mathbf{b}}{2}+\frac{1}{4}(\mathbf{c}+\mathbf{a})$

$$=\tfrac{1}{4}(\mathbf{a}+2\mathbf{b}+\mathbf{c}).$$

Now if O divides CR in the ratio $3:1$, then the position vector of O should be $\dfrac{3.\dfrac{2\mathbf{b}+\mathbf{a}}{3}+1.\mathbf{c}}{3+1}$ or $\frac{1}{4}(\mathbf{a}+2\mathbf{b}+\mathbf{c})$ which is true.

Again equation to AO is $\mathbf{r}=(1-t)\mathbf{a}+t.\frac{1}{4}(\mathbf{a}+2\mathbf{b}+\mathbf{c})$.

Since they intersect at P we have on comparing,

$$1-t+\frac{t}{4}=0,\ \frac{t}{2}=1-s,\ \frac{t}{4}=s.$$

From first two we get $t=\frac{4}{3}.s=\frac{1}{3}$ and these satisfy the third relation also.

$\therefore$ position vector of P is $\dfrac{2\mathbf{b}+\mathbf{c}}{3}$.

If P divides BC in the ratio $m:1$, then position vector of P is

$$\frac{m\mathbf{c}+\mathbf{b}}{m+1}=\frac{2\mathbf{b}+\mathbf{c}}{3}.$$

Comparing, $\dfrac{m}{m+1}=\frac{1}{3}$ and $\dfrac{1}{m+1}=\frac{2}{3}$ $\quad\therefore\quad m=\frac{1}{2}$.

$\therefore$ P divides BC in the ratio $1:2$.

Ex. 30. *A line from a vertex of a triangle bisects the opposite side. It intersects a similar line issuing from the other vertex. Prove that these lines intersect in the ratio 3 : 1.*

Ex. 31. *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be non-coplanar, show that the lines whose equations are* $\mathbf{r}=(-10\mathbf{a}+\mathbf{b}-\mathbf{c})+t(8\mathbf{a}+2\mathbf{b}-3\mathbf{c})$,

$$\mathbf{r}=(-\mathbf{a}+4\mathbf{b}-3\mathbf{c})+s(7\mathbf{a}+\mathbf{b}-\mathbf{c})$$

are coplanar, and find their point of intersection.

If the above lines are coplanar, then they must intersect for which we should get identical values of $\mathbf{r}$. Comparing $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in the two equations we will have three relations in s and t. Solving the two we find that the

third is also satisfied and hence they intersect and the point of intersection is obtained as $6\mathbf{a}+5\mathbf{b}-7\mathbf{c}$.

Ex. 32. *Prove by vectorial methods the equation* $\frac{x}{a}+\frac{y}{b}=1$ *which is the equation of a straight line in terms of its intercepts on the axes of reference. The axes may be rectangular or oblique.*

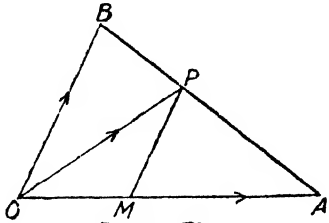


Fig No 50

Let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$ respectively and their modules be a and b, so that $\mathbf{a}=a.\hat{\mathbf{a}}$ and $\mathbf{b}=b.\hat{\mathbf{b}}$.

Let P be any point (x, y) on the line. From P draw PM parallel to OB, so that $OM=x$ and $MP=y$.

$$\therefore \quad \vec{OM}=x.\hat{a}=\frac{x}{a}.\mathbf{a} \text{ and } \vec{MP}=y.\hat{b}=\frac{y}{b}.\mathbf{b}.$$

Now P being any point on AB,

$$\vec{OP}=\mathbf{r}=(1-t)\,\mathbf{a}+t\mathbf{b} \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

Also $$\vec{OP}=\vec{OM}+\vec{MP}=\frac{x}{a}.\mathbf{a}+\frac{y}{b}.\mathbf{b}.. \quad \ldots \quad \ldots(2)$$

Comparing $\mathbf{a}$ and $\mathbf{b}$ in (1) and (2), we get

$$1-t=\frac{x}{a}, \quad t=\frac{y}{b}.$$

Eliminating t, we get $\frac{x}{a}+\frac{y}{b}=1$ as the required equation.

Ex. 33. *The straight line through the mid. points of three coplanar edges of a tetrahedron, each parallel to the line joining a fixed point, O to the mid. point of the opposite edge are concurrent at a point P such that OP is bisected by the centroid (of volume) of the tetrahedron.* **(Agra 46)**

Take O as origin and let the position vectors of A, B, C and D be **a, b, c, d** respectively.

Let P_1 and T_1 be the mid. points of opposite edges, so that their position vectors are

$$\frac{\mathbf{b}+\mathbf{c}}{2} \text{ and } \frac{\mathbf{a}+\mathbf{d}}{2}.$$

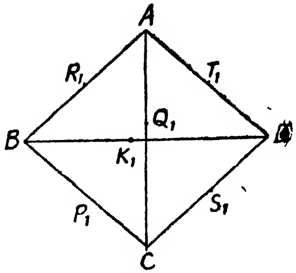


Fig. No 51

Now the equation of a line through P_1 and parallel to OT_1 is

$$\mathbf{r}=\frac{\mathbf{b}+\mathbf{c}}{2}+t\left(\frac{\mathbf{a}+\mathbf{d}}{2}\right) \quad \ldots\ldots\ldots\ldots(1)$$

Similarly we can write down the equation of line through S_1 *i. e.* $\frac{\mathbf{c}+\mathbf{d}}{2}$ and parallel to OR_1 where R_1 is $\frac{\mathbf{a}+\mathbf{b}}{2}$ as

$$r=\frac{\mathbf{c}+\mathbf{d}}{2}+\frac{s}{2}\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \quad \ldots\ldots\ldots\ldots(2)$$

The equation of the line through k_1 i. e. $\frac{\mathbf{b}+\mathbf{d}}{2}$ and parallel to OQ_1 where Q_1 is $\frac{\mathbf{a}+\mathbf{c}}{2}$ is

$$r=\frac{\mathbf{b}+\mathbf{d}}{2}+p\left(\frac{\mathbf{a}+\mathbf{c}}{2}\right) \quad \ldots\ldots\ldots\ldots(3)$$

Now (1) and (2) intersect at P and hence on comparing we have $t=s=1$ and we get the point P as $\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$.

Again if we choose $p=1$, the point satisfies (3) also.

Hence the above three lines are concurrent at P.

Now mid-point of OP is

$$\tfrac{1}{2}\left[0+\tfrac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})\right]=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$$

which is the centroid of volume G of the tetrahedron. Hence OP is bisected at G.

Ex. 54. *If* **a**, **b** *are two non-collinear vectors, show that the points* $l_1\mathbf{a}+m_1\mathbf{b}$, $l_2\mathbf{a}+m_2\mathbf{b}$, $l_3\mathbf{a}+m_3\mathbf{b}=0$ *are collinear if and only if*

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix}=0.$$

(Agra 42)

If the given vectors are collinear, then we must have

$$x\,(l_1\mathbf{a}+m_1\mathbf{b})+y\,(l_2\mathbf{a}+m_2\mathbf{b})+z\,(l_3\mathbf{a}+m_3\mathbf{b})=0.$$

where $x+y+z=0$... ... ...(1)

[§ 7. P. 48]

or $(xl_1+yl_2+zl_3)\,\mathbf{a}+(xm_1+ym_2+zm_2)\,\mathbf{b}=0.$

Since **a** and **b** are two non-collinear vectors and if there exists a relation of the type $l\mathbf{a}+m\mathbf{b}=0$, then $l=0$, $m=0$ [§ 5 chapter 1, P. 16]

or $xl_1+yl_2+zl_3=0$... ... ...(2)

$xm_1+ym_2+zm_3=0$... ... ...(3)

Eliminating x, y, z betwen (2). (3) and (1), we get the required condition.

§ 9. Vector equation of a plane.

To find the vector equation of a plane which passes through the origin and is parallel to two given vectors **a** *and* **b**.

Choosing O as origin let the vectors OA and OB be **a** and **b**. Let P be any point on the plane. From P draw PM and PN parallel to OB and OA respectively meeting OA and OB in M and N respectively.

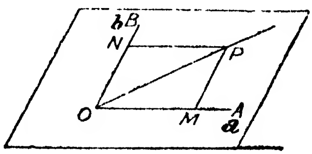


Fig. No. 52

$\overrightarrow{OM}$ being collinear with $OA=s\mathbf{a}$.

$\overrightarrow{ON}$ being collinear with $OB = t\mathbf{b}$.

If $\mathbf{r}$ be the position vector of P any point on the plane, then $\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \overrightarrow{OM} + \overrightarrow{ON} = s\mathbf{a} + t\mathbf{b}$.

Hence the vector equation of the required plane is given by

$$r = s\mathbf{a} + t\mathbf{b} \quad \ldots \quad \ldots \quad \ldots \quad \ldots (1)$$

where s and t are scalars which vary as the point P moves on the plane.

Note. See § 5 P. 16 Chapter I.

Cor. 1. *To find the vector equation of a plane through a given point* **c** *and parallel to* **a** and **b**. **[Agra 40]**

Let **c** be the position vector of any point C on the plane and P be and point on it. Now the vector $\overrightarrow{CP}$ is coplanar with **a** and **b** and as such $\overrightarrow{CP} = s\mathbf{a} + t\mathbf{b}$.

(§ 5 chapter I P. 15).

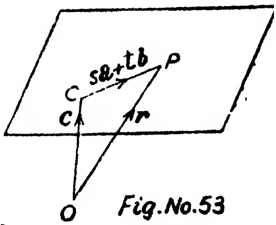


Fig.No.53

If $\mathbf{r}$ be the position vector of P, then

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP} = \mathbf{c} + s\mathbf{a} + t\mathbf{b}. \quad \ldots \quad \ldots \quad \ldots (1)$$

Hence the required vector equation of the plane is $\mathbf{r} = \mathbf{c} + s\mathbf{a} + t\mathbf{b}$ where s and t are scalars which vary as the point P moves on the plane.

Cor. 2. *To find the vector equation of a plane that passes through three points whose position vectors are* **a**, **b** *and* **c**.

(Agra 39)

Let **a**, **b**, **c** be the position vectors of the points A, B and C respectively on the plane, so that $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and

$\vec{AC}=\mathbf{c}-\mathbf{a}$.

Now the required plane is one through the point **a** and parallel to $\vec{AB}$ and $\vec{AC}$, *i.e.* through **a** and parallel to $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ and hence by Cor. 1, its equation is

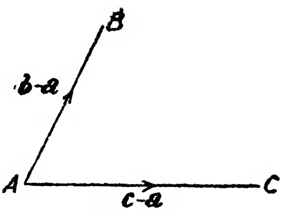


Fig. No. 54

$$r=\mathbf{a}+s(\mathbf{b}-\mathbf{a})+t(\mathbf{c}-\mathbf{a})$$

or $$r=(1-s-t)\,\mathbf{a}+s\mathbf{b}+t\mathbf{c} \quad \ldots \quad \ldots \quad \ldots (3)$$

Cor. 3. *To find the vector equation of a plane through two points* **a** *and* **b** *and parallel to* **c**.

If **a** and **b** be the position vectors of A and B respectively, then $\vec{AB}=\mathbf{b}-\mathbf{a}$. Hence the required plane is one that passes through **a** and is parallel to $\mathbf{b}-\mathbf{a}$ and **c** and hence by Cor. 1, its equation is

$$\mathbf{r}=\mathbf{a}+s(\mathbf{b}-\mathbf{a})+t\mathbf{c}$$

or $$\mathbf{r}=(1-s)\,\mathbf{a}+s\mathbf{b}+t\mathbf{c} \quad \ldots \quad \ldots \quad \ldots (4)$$

Cor. 4. *To find the Cartesian equivalents of the vector equations of planes found above.*

[Refer author's Solid Geometry.]

Case 1. Plane through origin and parallel to the given line.

Let in terms of unit vectors **i**, **j** and **k** through origin O, the vector

$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ so that P is (x, y, z).

$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ so that direction cosines of vector **a** are proportional to a_1, a_2, a_3,

and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ so that direction cosines of vector **b** are proportional to b_1, b_2, b_3.

(a) Now vector equation of a plane through origin

and parallel to vectors **a** and **b** is

$$\mathbf{r}=s\mathbf{a}+t\mathbf{b}$$

or $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=s(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})+t(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$.

Equating coefficients of **i**, **j** and **k**, we get

$$x-sa_1-tb_1=0,$$
$$y-sa_2-tb_2=0,$$
$$z-sa_3-tb_3=0.$$

Eliminating $-s$ and $-t$, we get

$$\begin{vmatrix} x & a_1 & b_1 \\ y & a_2 & b_2 \\ z & a_3 & b_3 \end{vmatrix}=0.$$

or $x(a_2b_3-a_3b_2)+y(a_3b_1-a_1b_3)+z(a_1b_2-a_2b_1)=0.$

Above is the corresponding cartesian equation of a plane through origin and parallel to lines whose direction cosines are proportional to a_1, a_2, a_3 and b_1, b_2, b_3.

(b) Case 2. Plane through a given point and parallel to two given lines.

Vector equation of a plane through a point and parallel to two straight lines is

$$\mathbf{r}=\mathbf{c}+s\mathbf{a}+t\mathbf{b} \qquad \text{[by Cor. 1]}$$

or $$(\mathbf{r}-\mathbf{c})=s\mathbf{a}+t\mathbf{b}.$$

Putting in terms of unit vectors and equating coefficients of **i**, **j** and **k**, we get

$$(x-c_1)-sa_1-tb_1=0$$
$$(y-c_2)-sa_2-tb_2=0.$$
$$(z-c_3)-sa_3-tb_3=0.$$

Eliminating $-s$, $-t$, we get

$$\begin{vmatrix} x-c_1 & a_1 & b_1 \\ y-c_2 & a_2 & b_2 \\ z-c_3 & a_3 & b_3 \end{vmatrix}=0$$

or
$$\begin{vmatrix} x & a_1 & b_1 \\ y & a_2 & b_2 \\ z & a_3 & b_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}.$$

Above is the corresponding cartesian equation of a plane that passes through the point (c_1, c_2, c_2) and is parallel to two lines whose direction cosines are proportional to a_1, a_2, a_3 and $b_1, b_2\ b_3$.

(c) Case 3. Plane passing through three given points.

Vector equation of a plane through three given points is

$$\mathbf{r}=(1-s-t)\,\mathbf{a}+s\mathbf{b}+t\mathbf{c} \quad \textbf{[Cor. 2]}$$

or
$$\mathbf{r}=\mathbf{a}+s\,(\mathbf{b}-\mathbf{a})+t\,(\mathbf{c}-\mathbf{a}).$$

Putting in terms of unit vectors and equating coefficients of **i**, **j** and **k**, we get

$$(x-a_1)-s\,(b_1-a_1)-t\,(c_1-a_1)=0,$$
$$(y-a_2)-s\,(b_2-a_2)-t\,(c_2-a_2)=0,$$
$$(z-a_3)-s\,(b_3-a_3)-t\,(c_3-a_3)=0.$$

Eliminating $-s$ and $-t$ between the above equations, we get

$$\begin{vmatrix} x-a_1 & b_1-a_1 & c_1-a_1 \\ y-a_2 & b_2-a_2 & c_2-a_2 \\ z-a_3 & b_3-a_3 & c_3-a_3 \end{vmatrix}=0.$$

The above determinant can be written as a fourth order determinant as following :

$$\begin{vmatrix} x-a_1 & b_1-a_1 & c_1-a_1 & a_1 \\ y-a_2 & b_2-a_2 & c_2-a_2 & a_2 \\ z-a_3 & b_3-a_3 & c_3-a_3 & a_3 \\ 0 & 0 & 0 & 1 \end{vmatrix}=0$$

Adding fourth column to first, second and third column, we get

$$\begin{vmatrix} x & b_1 & c_1 & a_1 \\ y & b_2 & c_2 & a_2 \\ z & b_3 & c_3 & a_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0 \text{ or } \begin{vmatrix} x & y & z & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ a_1 & a_2 & a_3 & 1 \end{vmatrix}=0$$

or
$$\begin{vmatrix} x & y & z & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{vmatrix}=0.$$

Above is the corresponding cartesian equation of a plane through three points (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3).

§ 10. Condition for four points to be coplanar.

To prove that the necessary and sufficient condition for any four points in three-dimensional space to be coplanar is that there exists a linear relation connecting their position vectors such that the algebraic sum of the coefficients in it is zero.

(Pb. 60; Agra 37, 54)

We have seen that the vector equation of a plane through the points whose position vectors are **a**, **b** and **c** is

$$\mathbf{r}=(1-s-t)\,\mathbf{a}+s\mathbf{b}+t\mathbf{c}=0. \qquad \textbf{[Cor. 2]}$$

The above relation may be written as

$$(1-s-t)\,\mathbf{a}+s\mathbf{b}+t\mathbf{c}-\mathbf{r}=0.$$

Above is a relation between the position vectors of four points A, B, C and D which are coplanar and we observe that algebraic sum of the coefficients of **a**, **b**, **c** and **r** is $1-s-t+s+t-1$ which is zero. Hence the condition is necessary.

In order to prove that the condition is sufficient let us suppose that any four vectors **a**, **b**, **c**, **d** be connected by the relation $l\mathbf{a}+m\mathbf{b}+n\mathbf{c}+p\mathbf{d}=0$ where $l+m+n+p=0$.

Dividing by p ($p\neq 0$), we get

$$\mathbf{d}=-\frac{l}{p}\mathbf{a}-\frac{m}{p}\mathbf{b}-\frac{n}{p}\mathbf{c},$$

where
$$\frac{l}{p}+\frac{m}{p}+\frac{n}{p}=-1.$$

Putting $-\dfrac{m}{p}=s$ and $-\dfrac{n}{p}=t$,

$$\frac{l}{p}=-1-\frac{m}{p}-\frac{n}{p}=-1+s+t.$$

$$\therefore\quad \mathbf{d}=(1-s-t)\,\mathbf{a}+s\mathbf{b}+t\mathbf{c}.$$

Above relation shows that **d** is a point on the plane through **a**, **b** and **c**; hence **a**, **b**, **c**, **d** are coplanar.

Alternative Proof.

Let us suppose that the points A, B, C, D whose position vectors are **a**, **b**, **c**, **d** are coplanar and let AB and CD intersect at P (it being assumed that AB and CD are not parallel and if they are, then we will choose any other pair of non-parallel lines formed by given points).

If P divides AB in the ratio $p:q$ and CD in $m:n$, then its position vector written from AB and CD is

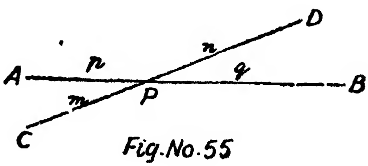


Fig. No. 55

$$\frac{p\mathbf{b}+q\mathbf{a}}{p+q}=\frac{m\mathbf{d}+n\mathbf{c}}{m+n}$$

or
$$\frac{q}{p+q}\mathbf{a}+\frac{p}{p+q}\mathbf{b}-\frac{n}{m+n}\mathbf{c}-\frac{m}{m+n}\mathbf{d}=0$$

or
$$L\mathbf{a}+M\mathbf{b}+N\mathbf{c}+P\mathbf{d}=0$$

where $L+M+N+P=\dfrac{q}{p+q}+\dfrac{p}{p+q}-\dfrac{n}{m+n}-\dfrac{m}{m+n}=1-1=0.$

Hence the condition is necessary.

Note. From here we find the position vector of any point **d** on the plane through **a**, **b**, **c** as

$$\mathbf{d}=\frac{L\mathbf{a}+M\mathbf{b}+N\mathbf{c}}{-P}=\frac{L\mathbf{a}+M\mathbf{b}+N\mathbf{c}}{L+M+N}.$$

$$\because \quad L+M+N+P=0.$$

Converse.

Again let $l\mathbf{a}+m\mathbf{b}+n\mathbf{c}+p\mathbf{d}=0$ where $l+m+n+p=0$ and we will show that the points A, B, C, D are coplanar.

Now of the three scalars $l+m$, $l+n$, $l+p$, one at least is not zero for if all of them were zero then

$$l+m=0,\ l+n=0,\ l+p=0.$$

$$\therefore \quad m-n=0,\ n-p=0 \text{ or } m=n=p=-l.$$

Also $\quad l+m+n+p=0$ but $l+m=0$.

$$\therefore \quad n+p=0 \text{ or } 2p=0; \quad \therefore \quad p=0.$$

$\therefore \quad l=0=m=n=p$ which is impossible.

Let us suppose that $l+m$ is not zero and therefore

$$l+m=-(n+p)\neq 0.$$

From the given relation, we get

$$(l\mathbf{a}+m\mathbf{b})=-(n\mathbf{c}+p\mathbf{d})$$

or

$$\frac{l\mathbf{a}+m\mathbf{b}}{l+m}=\frac{n\mathbf{c}+p\mathbf{d}}{n+p}.$$

L. H. S. is centroid of points A and B with associated numbers l and m and hence is on AB, and R. H. S. is centroid of C and D with associated numbers n and p, hence is on CD. Hence we find a point on AB is same as a point on CD showing that they intersect and hence A, B, C, D are coplanar.

Also the point of intersection of AB and CD is

$$\frac{l\mathbf{a}+m\mathbf{b}}{l+m} \text{ or } \frac{n\mathbf{c}+p\mathbf{d}}{n+p}.$$

Exercise

Ex. 1. *If any point within a tetrahedron ABCD is joined to the vertices and AO, BO, CO, DO are produced to cut the opposite faces in P, Q, R, S, show that*

$$\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}+\frac{OS}{DS}=1.$$

(Agra 33, 35, 39, 43, 46, 49, 58, 61)

Taking O as origin let the position vectors of A, B, C, D be **a**, **b**, **c**, **d** respectively and we know that any four vectors are linearly dependent *i.e.* there exists a relation of the form

$$l\mathbf{a}+m\mathbf{b}+n\mathbf{c}+p\mathbf{d}=0 \quad \ldots(1)$$

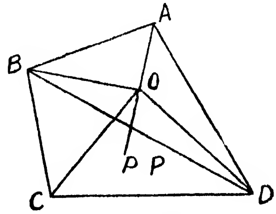


Fig No 56

Equation of $\overrightarrow{AO}$ is $\mathbf{r}=-\mathbf{a}$ and hence equation to OP is $r=-k\mathbf{a}$ where k is a scalar positive for points on AO produced and $-$ ive for those on OA.

But from (1), we have

$$\mathbf{a}=-\frac{m\mathbf{b}+n\mathbf{c}+p\mathbf{d}}{l}.$$

Putting for **a** from above the equation to $\overrightarrow{OP}$ is given by

$$\mathbf{r}=\frac{k}{l}(m\mathbf{b}+n\mathbf{c}+p\mathbf{d}) \quad \ldots \quad \ldots(2)$$

Again equation to plane BCD is

$$\mathbf{r}=(1-s-t)\,\mathbf{b}+s\mathbf{c}+t\mathbf{d} \quad \textbf{[Cor. 2]} \quad \ldots \quad \ldots(3)$$

Now (2) and (3) intersect at P and hence we have on comparing,

$$\frac{km}{l}=1-s-t,\ \frac{kn}{l}=s,\ \frac{kp}{l}=t.$$

Adding, $\frac{k}{l}(m+n+p)=1$ or $k=\frac{l}{m+n+p}$ ... ...(iv)

Hence position vector of P is $-\frac{l}{m+n+p}\,\mathbf{a}$.

Again $\overrightarrow{AP}=\overrightarrow{AO}+\overrightarrow{OP}=-\mathbf{a}-\frac{l}{m+n+p}\,\mathbf{a}=-\frac{l+m+n+p}{m+n+p}\,\mathbf{a}$

or $$\overrightarrow{AP}=-\frac{l+m+n+p}{l}\,k\mathbf{a}=\frac{l+m+n+p}{l}\overrightarrow{OP}\ \text{[by (iv)]}$$

$$\because\ \overrightarrow{OP}=-k\mathbf{a}.$$

$\therefore\ \frac{OP}{AP}=\frac{l}{l+m+n+p}$. Proceeding as above, we have

$$\frac{OQ}{BQ}=\frac{m}{l+m+n+p},\ \frac{OR}{CR}=\frac{n}{l+m+n+p},$$

and $$\frac{OS}{DS}=\frac{p}{l+m+n+p}.$$

Adding, we get the required result.

2. *Find the equation of the plane through the origin and the points* $4\mathbf{j}$ *and* $2\mathbf{i}+\mathbf{k}$. *Find also the point in which this plane is cut by the line joining the points* $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ *and* $3\mathbf{k}-2\mathbf{j}$.

(Agra 42, 45, 56)

Equation of plane is $\mathbf{r}=s\,.\,4\mathbf{j}+t\,(2\mathbf{i}+\mathbf{k})$.

Equation of line is $\mathbf{r}=(1-p)\,(\mathbf{i}-2\mathbf{j}+\mathbf{k})+p\,(3\mathbf{k}-2\mathbf{j})$.

For point of intersection, comparing and solving, we find that $t=\frac{3}{5}$, $s=-\frac{1}{2}$ and $p=-\frac{1}{5}$, and hence the point is $\frac{1}{5}\,(6\mathbf{i}-10\mathbf{j}+3\mathbf{k})$.

3. *Prove that the four points* $2\mathbf{a}+3\mathbf{b}-\mathbf{c}$, $\mathbf{a}-2\mathbf{b}+3\mathbf{c}$, $3\mathbf{a}+4\mathbf{b}-2\mathbf{c}$ *and* $\mathbf{a}-6\mathbf{b}+6\mathbf{c}$ *are coplanar.*

1st Method. Let the given points be A, B, C, D respectively. We will prove that the lines joining any two intersects the line joining the other two provided they are not parallel and hence the four points are coplanar.

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\mathbf{a}-5\mathbf{b}+4\mathbf{c},$$

$$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=-2\mathbf{a}-10\mathbf{b}+8\mathbf{c}.$$

We find that $\overrightarrow{AB}=-2\overrightarrow{CD}$ *i. e.* AB is parallel to CD.
∴ let us consider the lines AC and BD say.

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\mathbf{a}+\mathbf{b}-\mathbf{c},$$

$$\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=-4\mathbf{b}+3\mathbf{c}.$$

Equation to AC is $\mathbf{r}=2\mathbf{a}+3\mathbf{b}-\mathbf{c}+t\,(\mathbf{a}+\mathbf{b}-\mathbf{c})$.
Equation to BD is $\mathbf{r}=\mathbf{a}-2\mathbf{b}+3\mathbf{c}+s\,(-4\mathbf{b}+3\mathbf{c})$.
If they intersect, we have on comparing,

$$2+t=1,\ 3+t=-2-4s,\ -1-t=3+3s.$$

Solving the first two, we find $t=-1$ and $s=-1$ and these values satisfy the 3rd equation as well. Hence the two lines intersect showing thereby that the four points are coplanar and the point of intersection is $\mathbf{a}+2\mathbf{b}$.

2nd Method. $\overrightarrow{AB}=-\mathbf{a}-5\mathbf{b}+4\mathbf{c},$

$$\overrightarrow{AC}=\mathbf{a}+\mathbf{b}-\mathbf{c},$$

and $$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=-\mathbf{a}-9\mathbf{b}+7\mathbf{c}.$$

If the given points are coplanar, then the three coterminous vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ should be coplanar and as such they should be linearly dependent, *i. e.*, there must exist a relation between them. Let us suppose that

$$l\,.\,\overrightarrow{AB}+m\,.\,\overrightarrow{AC}=\overrightarrow{AD}$$

or $$l\,(-\mathbf{a}-5\mathbf{b}+4\mathbf{c})+m\,(\mathbf{a}+\mathbf{b}-\mathbf{c})=-\mathbf{a}-9\mathbf{b}+7\mathbf{c}.$$

Comparing, we get

$$-l+m=-1,\ -5l+m=-9,\ 4l-m=7.$$

Solving the first two, we get $l=2$, $m=1$ and these values satisfy the third relation $4l-m=7$, and hence $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar. Therefore the points A, B, C, D are coplanar.

4. *Prove that the points* $6\mathbf{a}+2\mathbf{b}-\mathbf{c}$, $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$, $-\mathbf{a}+2\mathbf{b}-4\mathbf{c}$ *and* $-12\mathbf{a}-\mathbf{b}-3\mathbf{c}$ *are coplanar.*

5. *Prove that the points* $-\mathbf{a}+4\mathbf{b}-3\mathbf{c}$, $3\mathbf{a}+2\mathbf{b}-5\mathbf{c}$, $3\mathbf{a}+8\mathbf{b}-5\mathbf{c}$, $-3\mathbf{a}+2\mathbf{b}+\mathbf{c}$ *are coplanar.*

6. (a) *Find the point in which the plane*

$$\mathbf{r}=\mathbf{a}-\mathbf{b}+s\,(\mathbf{a}+\mathbf{b}-\mathbf{c})+t\,(\mathbf{a}+\mathbf{c}-\mathbf{b})$$

is cut by the line through the point $2\mathbf{a}+3\mathbf{b}$ *and parallel to* $\mathbf{c}$.

The equation to the line is $\mathbf{r}=2\mathbf{a}+3\mathbf{b}+p\mathbf{c}$. For point of intersection compare etc. **Ans.** $2\mathbf{a}+3\mathbf{b}-4\mathbf{c}$

(b) *Find the intersection of the line joining the points* $(1, -2, -1)$ *and* $(2, 3, 1)$ *with the plane through the points* $(2, 1, -3)$, $(4, -1, 2)$ *and* $(3, 0, 1)$. **Ans.** $(\frac{5}{3}, \frac{4}{3}, \frac{1}{3})$

7. *Prove that the six planes containing one edge and bisecting the opposite edge of a tetrahedron bisect each other.*

Plane containing OA and bisecting BC *i.e.* plane OAD is

$$\mathbf{r}=s_1\mathbf{a}+t_1\left(\frac{\mathbf{b}+\mathbf{c}}{2}\right) \text{ [§ 9]}$$

Plane OCF is

$$\mathbf{r}=s_2\mathbf{c}+t_2\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \text{ [§ 9]}.$$

Plane PBC is

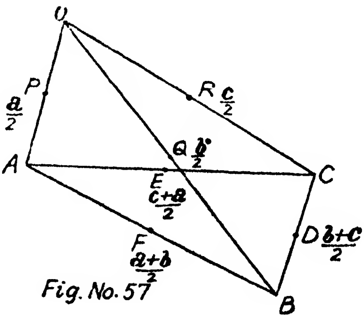


Fig. No. 57

$$\mathbf{r}=(1-s_3-t_3)\,\frac{\mathbf{a}}{2}+s_3\mathbf{b}+t_3\mathbf{c} \text{ [§ 9 Cor. 1]}.$$

Compare the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ for point of intersection.

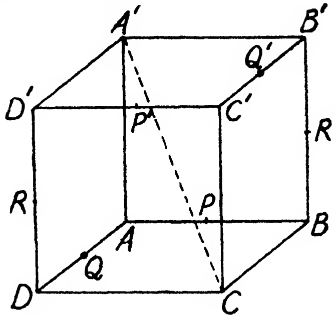
$$s_1=\frac{t_2}{2}=\frac{1-s_3-t_3}{2}, \quad \frac{t_1}{2}=\frac{t_2}{2}=s_3, \quad \frac{t_1}{2}=s_2=t_3.$$

$$\therefore \quad t_1=t_2=\tfrac{1}{2} \quad \text{and} \quad s_1=\tfrac{1}{4}.$$

Hence the point of intersection is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{4}$.

Similarly we can show for other planes.

7. (b) *Prove that the middle points of the six edges of a cube which do not meet a particular diagonal are coplanar.*



Let A be taken as origin and AB, AD and AA' as axes along which the unit vectors be $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively. Let the edge of the cube be of unit length. P, Q, R and P', Q', R' are the mid. points of the edges on which they are marked. The position vector of these mid. points are

$$P\ \frac{\mathbf{i}}{2},\ Q\ \frac{\mathbf{j}}{2},\quad R\ \frac{\mathbf{j}+\mathbf{j}+\mathbf{k}}{2}=\mathbf{j}+\frac{\mathbf{k}}{2},\ C'\ \mathbf{i}+\mathbf{j}+\mathbf{k},$$

$$P'\ \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})+(\mathbf{j}+\mathbf{k})}{2}=\frac{\mathbf{i}}{2}+\mathbf{j}+\mathbf{k}.$$

Similarly, Q' is $\mathbf{i}+\dfrac{\mathbf{j}}{2}+\mathbf{k}$

and $$R'\ \frac{\mathbf{i}+\mathbf{i}+\mathbf{k}}{2}=\mathbf{i}+\frac{\mathbf{k}}{2}.$$

Now the equation to the plane PQR is

$$\mathbf{r}=(1-s-t)\,\frac{\mathbf{i}}{2}+s\,.\,\frac{\mathbf{j}}{2}+t\left(\mathbf{j}+\frac{\mathbf{k}}{2}\right). \qquad \ldots(1)$$

The point $P'\ \dfrac{\mathbf{i}}{2}+\mathbf{j}+\mathbf{k}$ will lie on it, if on comparing the coefficients,

$$1-s-t=1, \quad \frac{s}{2}+t=1, \quad \frac{t}{2}=1.$$

Solving the last two, we get $t=2, s=-2$, and these values satisfy the first also.

Similarly, we can show that for some suitable values of s and t the other two mid. points Q' and R' will also lie on the same plane. Hence P, Q, R, P', Q' R' are coplanar.

8. *If a, b, c be any three* **non-coplanar vectors,** *then prove that the points* $l_1\mathbf{a}+m_1\mathbf{b}+n_1\mathbf{c}$, $l_2\mathbf{a}+m_2\mathbf{b}+n_2\mathbf{c}$, $l_3\mathbf{a}+m_3\mathbf{b}+n_3\mathbf{c}$, $l_4\mathbf{a}+m_4\mathbf{b}+n_4\mathbf{c}$ *are coplanar if and only if*

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0.$$

If the given vectors are coplanar, then there must exist a relation of the form

$$x(l_1\mathbf{a}+m_1\mathbf{b}+n_1\mathbf{c})+y(l_2\mathbf{a}+m_2\mathbf{b}+n_2\mathbf{c})+z(l_3\mathbf{a}+m_3\mathbf{b}+n_3\mathbf{c}) + w(l_4\mathbf{a}+m_4\mathbf{b}+n_4\mathbf{c})=0,$$

where $$x+y+z+w=0. \qquad \ldots(1)$$

$$\therefore \quad xl_1+yl_2+zl_3+wl_4)\,\mathbf{a}+(xm_1+ym_2+zm_3+wm_4)\,\mathbf{b} +(xn_1+yn_2+zn_3+wn_4)\,\mathbf{c}=0.$$

Since **a**, **b**, **c** are three **non-coplanar vectors** and if there exists a relation of the form $l\mathbf{a}+m\mathbf{b}+n\mathbf{c}=0$, then

$$l=m=n=0. \qquad [\S\ 5 \text{ Chapter 1, P. } 15]$$

$$\therefore \quad xl_1+yl_2+zl_3+wl_4=0. \qquad \ldots(2)$$

$$xm_1+ym_2+zm_3+wm_4=0, \qquad \ldots(3)$$

$$xn_1+yn_2+zn_3+wn_4=0. \qquad \ldots(4)$$

Eliminating x, y, z and w between (2), (3), (4) and (1), we get the required result.

Therom of Ceva

9. *If P, Q, R are three points on the sides BC, CA, AB respectively of a triangle ABC, such that the lines AP, BQ ınl*

CR are concurrent, then $\dfrac{BP}{CP}\cdot\dfrac{CQ}{AQ}\cdot\dfrac{AR}{BR}=-1$ *and conversely.*

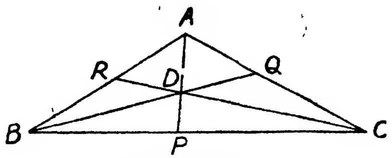


Fig. No. 58

Let D be the point of intersection of AP, BQ and the position vectors of A, B, C, D be $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ respectively. Since these four points are coplanar,

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+w\mathbf{d}=0 \text{ where } x+y+z+w=0\,;$$

$$\therefore\quad x\mathbf{a}+y\mathbf{b}=-(z\mathbf{c}+w\mathbf{d})$$

and $$x+y=-(z+w).$$

$$\therefore\quad \frac{x\mathbf{a}+y\mathbf{b}}{x+y}=\frac{z\mathbf{c}+w\mathbf{d}}{z+w}.$$

The L. H. S. is a point on AB and R. H. S. is a point on CD and both being same, either gives the point common to AB and CD and hence R.

Thus position vector of R is $\dfrac{x\mathbf{a}+y\mathbf{b}}{x+y}$ which is the centroid of A and B with associated numbers x and y, and thus divides AB in the ratio $y:x$.

$$\therefore\quad \frac{AR}{RB}=\frac{y}{x} \quad\text{or}\quad \frac{AR}{BR}=-\frac{y}{x}.$$

Similarly $\dfrac{BP}{CP}=-\dfrac{z}{y}$ and $\dfrac{CQ}{AQ}=-\dfrac{x}{z}$.

$$\therefore\quad \frac{AR}{BR}\cdot\frac{BP}{CP}\cdot\frac{CQ}{AQ}=-\frac{y}{x}\cdot-\frac{z}{y}\cdot-\frac{x}{z}=-1.$$

Hence proved.

Converse. We are given that

$$\frac{AR}{BR}\cdot\frac{BP}{CP}\cdot\frac{CQ}{AQ}=1 \qquad \ldots \quad \ldots \quad \ldots(1)$$

and we are to prove that AP, BQ and CR are concurrent.

Let us assume that $\frac{BP}{CP}=-\frac{z}{y}$ and $\frac{CQ}{AQ}=-\frac{x}{z}$ and hence from (1), $\frac{AR}{BR}=-\frac{y}{x}$.

From above, we get

$$\frac{BP}{PC}=\frac{z}{y},\ \frac{CQ}{QA}=\frac{x}{z},\ \frac{AR}{RB}=\frac{y}{x};$$

$$\therefore\quad P \text{ is } \frac{y\mathbf{b}+z\mathbf{c}}{y+z},\ Q \text{ is } \frac{z\mathbf{c}+x\mathbf{a}}{z+x},\ R \text{ is } \frac{x\mathbf{a}+y\mathbf{b}}{x+y}.$$

Suppose D is a point on AP dividing it in the ratio $y+z : x$ and therefore its position vector is

$$\frac{(y+z)\dfrac{y\mathbf{b}+z\mathbf{c}}{y+z}+x\mathbf{a}}{x+y+z}$$

or

$$\frac{x\mathbf{a}+y\mathbf{b}+z\mathbf{c}}{x+y+z}.$$

The symmetry of the result shows that this point will also lie on BQ dividing it in the ratio $z+x : y$ and on CR dividing it in the ratio $x+y : z$.

Hence the three lines are concurrent.

Theorem of Menelaus

10. *If P, Q, R are three points in the sides BC, CA and AB respectively of a triangle ABC, such that the points P, Q, R are collinear, then prove that* $\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1$ *and conversely.*

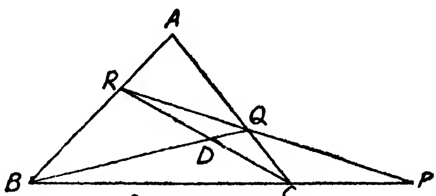


Fig. No. 59

Let D be the point of intersection of BQ and CR and the position vectors of A, B, C, D be taken as $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ respectively.

Since these four points are coplanar, we have

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+w\mathbf{d}=0 \text{ where } x+y+z+w=0.$$

Just as in Q. 9, the position vectors of Q and R on AC and AB respectively can be taken as

$$\mathbf{q}=\frac{x\mathbf{a}+z\mathbf{c}}{x+z};\quad \therefore\quad (x+z)\,\mathbf{q}=x\mathbf{a}+z\mathbf{c}\qquad \ldots\qquad \ldots(1)$$

$$\mathbf{r}=\frac{x\mathbf{a}+y\mathbf{b}}{x+y};\quad \therefore\quad (x+y)\,\mathbf{r}=x\mathbf{a}+y\mathbf{b}\qquad \ldots\qquad \ldots(2)$$

Subtracting, we get $(x+z)\,\mathbf{q}-(x+y)\,\mathbf{r}=z\mathbf{c}-y\mathbf{b}$.

Dividing both sides by $z-y$, we get

$$\frac{(x+z)\,\mathbf{q}-(x+y)\,\mathbf{r}}{(x+z)-(x+y)}=\frac{z\mathbf{c}-y\mathbf{b}}{z-y}.$$

L. H. S. gives a point on the line joining points whose position vectors are $\mathbf{q}$ and $\mathbf{r}$ *i. e.* on RQ and R. H. S. gives a point on the line joining $\mathbf{b}$ and $\mathbf{c}$, *i. e.* BC. Since they are equal therefore L. H. S. or R. H. S. gives the position vector of the point of intersection of BC and RQ, *i. e.* of P.

$$\therefore\quad \mathbf{p}=\frac{z\mathbf{c}-y\mathbf{b}}{z-y}.\qquad \ldots\qquad \ldots\qquad \ldots(3)$$

Now from (1), (2) and (3), we get

$$\frac{CQ}{QA}=\frac{x}{z},\ \frac{AR}{RB}=\frac{y}{x}\ \text{and}\ \frac{BP}{PC}=-\frac{z}{y}.$$

$$\therefore\quad \frac{BP}{CP}=\frac{z}{y},\ \frac{AR}{BR}=-\frac{y}{x},\ \frac{CQ}{AQ}=-\frac{x}{z}.$$

$$\therefore\quad \frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1.$$

Converse. We are given that

$$\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1.$$

Let us suppose that $\frac{BP}{CP}=\frac{z}{y}$ and $\frac{CQ}{AQ}=-\frac{x}{z}$, and hence $\frac{AR}{BR}=-\frac{y}{x}$.

$$\therefore \quad \frac{BP}{PC} = -\frac{z}{y}, \frac{CQ}{QA} = \frac{x}{z} \text{ and } \frac{AR}{RB} = \frac{y}{x}.$$

Hence if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are position vectors of P, Q and R respectively, then from above ratios,

$$\mathbf{p} = \frac{z\mathbf{c} - y\mathbf{b}}{z-y}, \; \mathbf{q} = \frac{x\mathbf{a} + z\mathbf{c}}{x+z}, \; \mathbf{r} = \frac{x\mathbf{a} + y\mathbf{b}}{x+y}.$$

Now if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are collinear then there must exist a relation of the form $L\mathbf{p} + M\mathbf{q} + N\mathbf{r} = 0$, such that

$$L + M + N = 0$$

$$-(z-y)\,\mathbf{p} + (x+z)\,\mathbf{q} - (x+y)\,\mathbf{r}$$
$$= -(z\mathbf{c} - y\mathbf{b}) + (x\mathbf{a} + z\mathbf{c}) - (x\mathbf{a} + y\mathbf{b}) = 0$$

where $\quad -(z-y) + (x+z) - (x+y) = 0.$ **Hence collinear.**

CHAPTER III

MULTIPLICATION OF VECTORS

§ 1. There are two different ways by which vector quantities are multiplied: one is called **scalar or dot product** and the other is called **vector or cross product.** The former is a mere number and does not involve any direction whereas the later is associated with a definite direction and as such is a vector quantity. However, in each case the product is proportional to the products of the lengths of the two vectors and they also follow the distributive law just as in the product of ordinary numbers. The scalar or dot product of two vectors **a** and **b** is written as **a . b** *i. e.* by placing a dot . between **a** and **b** whereas the vector or cross product of the vectors **a** and **b** is written as **a**×**b** *i. e.* by placing a cross × between **a** and **b**.

§ 2. **Scalar product.**

(Agra 32, 33, 40, 44, 51, 57, 58)

Def. *The scalar product of two vectors* **a** *and* **b** *of moduli a and* **b** *respectively is equal to ab cos θ where θ is the angle between the directions of* **a** *and* **b**.

Thus $\mathbf{a} \cdot \mathbf{b} = ab$ co $\theta = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta$.

Again $\mathbf{a} \cdot \mathbf{b} = a\,(b \cos\theta)$.

Now $b \cos\theta$ is length of the resolved part OL of OB *i. e.* module of vector **b** in the direction of **a** whose length is a or we can write it as

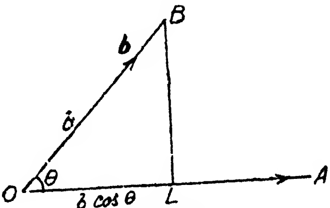


Fig No. 60

$$\mathbf{a} \cdot \mathbf{b} = b\,(a \cos\theta)$$

where $a \cos\theta$ is the length of the resolved part of **a** in the direction of **b** whose length is b.

Hence dot product of two vectors is equal to product of the length of one of them with resolved part of the other in the direction of the former.

Again if we take that **b** represents a force in magnitude and direction whereas **a** represents a vector drawn in an assigned direction, then $b \cos \theta$ is the resolved part of the force **b** in the direction of **a**. Thus $ab \cos \theta$ represents the work done by the force **b** in moving its point of application from O to A along OA and work does not involve the idea of a direction and hence dot product is a scalar quantity.

Properties of Dot product of Vectors

1. *From above we find that* $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = ab \cos \theta$.

Hence scalar product is commutative.

$$\therefore \quad \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} \quad \text{or} \quad \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|}.$$

2. *The dot product of two vectors will be +ive, zero or −ive according as* $\cos \theta$ *is +ive, zero or −ive which means that according as* θ *is acute or a right angle or obtuse.*

Again $\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$.

3. *Now* $\mathbf{a} \cdot \mathbf{b}$ *will be zero only if either* a *or* b *or* $\cos \theta$ *is zero.*

Hence dot product of two vectors is zero if either of the vectors be a zero vector or if they are non-zero vectors then **their directions should be perpendicular. [Remember].**

Again if θ be the angle between the directions of **a** and **b**, then $\pi - \theta$ will be the angle between the directions of **a** and $-\mathbf{b}$ or of $-\mathbf{a}$ and **b**

Thus $\mathbf{a} \cdot (-\mathbf{b}) = ab \cos (\pi - \theta) = (-\mathbf{a}) \cdot (\mathbf{b})$

or $$\mathbf{a} \cdot (-\mathbf{b}) = (-\mathbf{a}) \cdot \mathbf{b} = -ab \cos \theta = -(\mathbf{a} \cdot \mathbf{b}).$$

4. *In dot product of two vectors if either of the factors is*

multiplied by minus sign, then the total product is multiplied by minus sign.

Again if θ be the angle between the directions of **a** and **b**, then since vertically opposite angles are equal, θ is also the angle between the directions of $-$**a** and $-$**b**. Thus $(-\mathbf{a}).(-\mathbf{b})=ab\cos\theta=(\mathbf{a}.\mathbf{b})$. Again if a be the module of vector **a** then vector $m\mathbf{a}$ is a vector in the direction of **a** but of length ma.

Thus $(m\mathbf{a}).(n\mathbf{b})=(ma)(nb)\cos\theta=mn(ab\cos\theta)$

$$=mn(\mathbf{a}.\mathbf{b})$$

or $$=a.(mnb)\cos\theta=\mathbf{a}.(mn\mathbf{b})$$

or $$=(na)(mb)\cos\theta=n\mathbf{a}.m\mathbf{b}.$$

5. *Thus the scalar product of two vectors in associative.*

If the vectors be like, then angle between them is 0, *i. e.* $\cos\theta=1$.

$$\therefore\quad \mathbf{a}.\mathbf{b}=ab.$$

6. *Thus the product of two* **like** *vectors is equal to the product of their moduli.*

In case they be **unlike** then $\theta=180°$, *i. e.* $\cos\theta=-1$.

$$\therefore\quad \mathbf{a}.\mathbf{b}=-ab.$$

If the vectors be **equal** then $\mathbf{a}.\mathbf{a}=a.a.\cos 0=a^2$ and written as $\mathbf{a}^2=a^2$.

7. *Thus the square of any vector is equal to square of its module.* **(Remember)**

In case there be two unit vectors then their moduli are each unity. Thus $\hat{\mathbf{a}}.\hat{\mathbf{b}}=1.1.\cos\theta=\cos\theta$.

8. *Thus the product of two unit vectors is equal to the cosine of the angle between their directions.*

Orthonormal vector triads i, j, k.

We know that **i, j, k** are three mutually perpendicular unit vectors.

Donated by

$\therefore \quad \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1$ **(Properties 7)**

and $\quad \mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$ **(Property 3).**

9. Distributive Law *i. e.* $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.

(Punjab 60)

Taking O as origin let $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{BC}$ represent the vectors **a**, **b** and **c** respectively so that $\overrightarrow{OC}=\mathbf{b}+\mathbf{c}$.

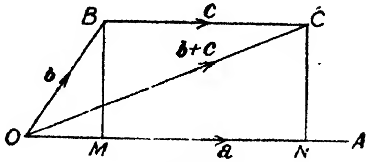


Fig No. 61

Again if BM and CN be perpendiculars from B and C on OA, then OM and ON are the projections of OB and OC on OA and hence MN is the projection of BC on OA.

Also $OM+MN=ON$.

Again we know that dot product of two vectors is equal to the product of the module of one with the resolved part *i. e.* projection of the other on the former.

$$\therefore \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot(\overrightarrow{OC})=a\cdot ON=a\,(OM+MN)$$

$$=a\cdot OM+a\cdot MN=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

Similarly $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=\mathbf{a}\cdot[\mathbf{b}+(-\mathbf{c})]=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot(-\mathbf{c})$

$$=\mathbf{a}\cdot\mathbf{b}+[-(\mathbf{a}\cdot\mathbf{c})]=\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}.$$

By repeated application of above, we obtain

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{b}$$

$$=\mathbf{a}^2-\mathbf{b}^2, \quad \because \quad \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}.$$

For geometrical interpretation of above see Q. 5 next exercise.

$$(\mathbf{a}+\mathbf{b})^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}$$

$$=\mathbf{a}^2+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}^2.$$

$$(\mathbf{a}-\mathbf{b})^2=\mathbf{a}^2-2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}^2.$$

In general $(\mathbf{a}+\mathbf{b}+\mathbf{c}+\ldots)\cdot(\mathbf{p}+\mathbf{q}+\mathbf{r}+\ldots)$

$$=\mathbf{a}\cdot\mathbf{p}+\mathbf{a}\cdot\mathbf{q}+\mathbf{a}\cdot\mathbf{r}$$

$$+\mathbf{b}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{q}+\mathbf{b}\cdot\mathbf{r}$$

$$+\mathbf{c}\cdot\mathbf{p}+\mathbf{c}\cdot\mathbf{q}+\mathbf{c}\cdot\mathbf{r}+\ldots$$

Again if **a** and **b** be expressed in terms of unit vectors

as $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\therefore$ $a=\sqrt{(a_1^2+a_2^2+a_3^2)}$;

and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+\mathbf{b}_3\mathbf{k}$, $\therefore$ $b=\sqrt{(b_1^2+b_2^2+b_3^2)}$;

then $\mathbf{a}\cdot\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$

or $$ab\cos\theta=a_1b_1+a_2b_2+a_3b_3.$$

$\because$ $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1$ and $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$. **(Prop. 8)**

or $$\cos\theta=\frac{a_1b_1+a_2b_2+a_3b_3}{ab}$$

$$=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{(a_1^2+a_2^2+a_3^2)}\cdot\sqrt{(b_1^2+b_2^2+b_3^2)}}\quad\ldots\ \ldots\ \ldots(1)$$

Note. (a_1, a_2, a_3) and (b_1, b_2, b_3) are the coordinates of the points A and B respectively.

Again if l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of **a** and **b**, then

$$l_1=\frac{a_1}{a},\ m_1=\frac{a_2}{a},\ n_1=\frac{a_3}{a}\text{ and } l_2=\frac{b_1}{b}\text{ etc.}$$

10. $\therefore$ $\cos\theta=l_1l_2+\mathbf{m}_1\mathbf{m}_2+\mathbf{n}_1\mathbf{n}_2$. [From (1)]

11. Components of a vector (Imp.).

(Delhi 50, Lucknow 52)

If any vector **r** is inclined at an angle θ to the direction of **a** and $\hat{\mathbf{a}}$ be a unit vector in this direction then the resolved part of **r** in the direction of **a** is $r\cos\phi\cdot\hat{\mathbf{a}}$

$$=\frac{ar\cos\phi}{a}\cdot\hat{\mathbf{a}}=\frac{(\mathbf{r}\cdot\mathbf{a})}{a^2}\,a\hat{\mathbf{a}}=\frac{(\mathbf{r}\cdot\mathbf{a})}{\mathbf{a}^2}\cdot\mathbf{a},\quad\because\quad a\hat{\mathbf{a}}=\mathbf{a}.$$

(Remember)

Again if **r** be resolved into two components in the plane of **a** and **r**, one parallel to **a** and the other perpendicular to **a**; then these components are

$$\frac{\mathbf{r}\cdot\mathbf{a}}{a^2}\cdot\mathbf{a}\text{ and }\mathbf{r}-\frac{\mathbf{r}\cdot\mathbf{a}}{a^2}\cdot\mathbf{a}.$$ You can write $\mathbf{a}^2$ for a^2

Similarly if a vector $\mathbf{r}$ be resolved into components parallel to unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, then since $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1$; these components are

$$(\mathbf{r}\cdot\mathbf{i})\,\mathbf{i},\ (\mathbf{r}\cdot\mathbf{j})\,\mathbf{j},\ (\mathbf{r}\cdot\mathbf{k})\,\mathbf{k}. \quad \textbf{(Remember)}$$

Alternative. We know that any vector $\mathbf{r}$ can be expressed in terms of three non-coplanar vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and let us suppose that $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$... ...(1) [§ 5 P. 15]

Mulplying successively by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and noting that

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0 \text{ and } \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1. \quad \textbf{(Property 8)}$$

$$\therefore \quad \mathbf{r}\cdot\mathbf{i}=x,\ \mathbf{r}\cdot\mathbf{j}=y,\ \mathbf{r}\cdot\mathbf{k}=z.$$

Putting the values of x, y and z in (1), we get

$$\mathbf{r}=(\mathbf{r}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{r}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{r}\cdot\mathbf{k})\,\mathbf{k}.$$

12. Again if $\mathbf{r}$ be expressed in terms of any three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then there exists a relation between them *i. e.* $\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$ ... ...(1)

Multiplying the above relation scalarly by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ successively, we get (§ 5 P. 15)

$$\mathbf{r}\cdot\mathbf{a}=x\mathbf{a}\cdot\mathbf{a}+y\mathbf{b}\cdot\mathbf{a}+z\mathbf{c}\cdot\mathbf{a} \quad \ldots \quad \ldots \quad \ldots(2)$$

$$\mathbf{r}\cdot\mathbf{b}=x\mathbf{a}\cdot\mathbf{b}+y\mathbf{b}\cdot\mathbf{c}+z\mathbf{c}\cdot\mathbf{b} \quad \ldots \quad \ldots \quad \ldots(3)$$

$$\mathbf{r}\cdot\mathbf{c}=x\mathbf{a}\cdot\mathbf{c}+y\mathbf{b}\cdot\mathbf{c}+z\mathbf{c}\cdot\mathbf{c} \quad \ldots \quad \ldots \quad \ldots(4)$$

Now $\mathbf{a}\cdot\mathbf{a}$, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{c}\cdot\mathbf{b}$ etc. are scalars. Hence eliminating x, y, z between (1), (2), (3) and (4), we get

$$\begin{vmatrix} \mathbf{r} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{r}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{r}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{r}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}=0.$$

Examples

Ex. 1. *Prove that in a tetrahedron if two pairs of opposite*

edges are perpendicular, the third pair are also perpendicular. Also show that sum of the squares on two opposite edges is the same for each pair. **(Lucknow 49, Agra 38, 53, Utkal 52)**

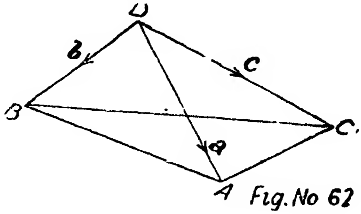


Fig. No 62

Taking D as origin, let the position vectors of points A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

We are given that AC is perpendicular to DB and AB is perpendicular to DC; and we are to prove that DA is perpendicular to BC.

$AC \perp$ to DB; $\therefore$ $\mathbf{b} \cdot (\mathbf{c}-\mathbf{a})=0$ or $\mathbf{b} \cdot \mathbf{c}=\mathbf{b} \cdot \mathbf{a}$...(1)

$\because$ **dot product of two perpendicular vectors is zero.**

$AB \perp$ to DC; $\therefore$ $\mathbf{c} \cdot (\mathbf{b}-\mathbf{a})=0$ or $\mathbf{c} \cdot \mathbf{b}=\mathbf{c} \cdot \mathbf{a}$. ...(2)

Now $\mathbf{a} \cdot \mathbf{b}=\mathbf{b} \cdot \mathbf{a}$; therefore from (1) and (2), we get

$\mathbf{a} \cdot \mathbf{b}=\mathbf{b} \cdot \mathbf{c}=\mathbf{c} \cdot \mathbf{a}$. ... ...(3)

From above we deduce that $\mathbf{a} \cdot (\mathbf{b}-\mathbf{c})=0$ showing that DA is perpendicular to CB. Hence proved.

Again $AB^2+CD^2=\vec{AB}^2+\vec{CD}^2=(\mathbf{b}-\mathbf{a})^2+(-\mathbf{c}^2)$,

$\because$ **square of a vector is equal to square of its module.** **(Property 7)**

$=\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-2\mathbf{a} \cdot \mathbf{b}$ which is symmetrical because $\mathbf{a} \cdot \mathbf{b}=\mathbf{b} \cdot \mathbf{c}=\mathbf{c} \cdot \mathbf{a}$ from (3).

$$\therefore \quad BD^2+CA^2=DA^2+BC^2=AB^2+CD^2.$$

Ex. 2. *Prove that sum of the squares on the edges of any tetrahedron*

(i) is equal to four times the sum of the squares on the joins of the mid-points of opposite edges. **(Agra 51)**

(ii) is equal to four times the sum of the squares of the lines joining the vertices to the centroids of opposite faces.

Ex. 3. *Prove that in right-angled triangle ABC,* $AB^2+AC^2=BC^2$, $\angle A$ *being a right angle.*

Taking A as origin and the position vectors of B and C as **b** and **c**, we have $\mathbf{b}\cdot\mathbf{c}=0$, $\because$ $\angle A=\pi/2$ etc.

Ex. 4. *Prove that the points* $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{i}-3\mathbf{j}-5\mathbf{k}$, $3\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ *are the vertices of a right-angled triangle.*

(Lucknow 52)

Find also the other two angles of the triangle.

Either calculate modules of each and prove that sum of the squares on the two is equal to square on third, or if the given points be A, B and C, then find $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ and prove that dot product of any two is zero.

Again $\mathbf{a}\cdot\mathbf{b}=ab\cos\theta$ etc. $\cos^{-1}\sqrt{(\frac{6}{41})}$, $\cos^{-1}\sqrt{(\frac{35}{41})}$.

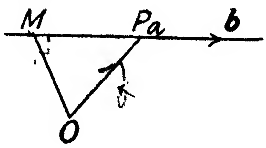
(Use Prop. 8)

(b) *If* $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ *be the equation of a st. line,* **b** *being a unit vector, prove that the line through origin perpendicular to it is* $\mathbf{r}=s\,\{\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}\}$ *and length of perpendicular is*

$\sqrt{\{\mathbf{a}^2-(\mathbf{a}\cdot\mathbf{b})^2\}}$. **(Agra 41)**

The given line passes through $P\mathbf{a}$ and is parallel to unit vector **b.** We are to find the equation to the line OM which passes through origin and is perpendicular to it.

Note : In the figure take OM perpendicular to MP.



MP=projection of OP **a** in the direction of unit vector $\mathbf{b}=a\cos\theta$.

But $\mathbf{a}\cdot\mathbf{b}=a\cdot1\cdot\cos\theta$.

$\therefore$ vector $\overrightarrow{MP}$ in the direction of unit vector **b** is of module $\mathbf{a}\cdot\mathbf{b}$.

$$\overrightarrow{MP}=(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}.$$

Now $$\vec{MP}=\vec{OP}-\vec{OM}$$

or $$\vec{OM}=\vec{OP}-\vec{MP}=\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}.$$

Hence the equation to OM *i. e.* a line through origin is

$$\mathbf{r}=s\,\{\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}\}.$$

Also $$OM^2=OP^2-MP^2.$$

$$\therefore\quad OM=\sqrt{\{\mathbf{a}^2-(\mathbf{a}\cdot\mathbf{b})^2\}}.$$

Ex. 5. *Prove that in any parallelogram the sum of the squares on the diagonals is twice the sum of the sauares on two adjacent sides; the difference of squares on the diagonals is four times the rectangle contained by either of these sides and projection of the other upon it; and the difference of the squares on two adjacent sides is equal to rectangle contained by either diagonal and the projection of the other upon it.*

Let $\vec{AB}=\mathbf{b}$, $\vec{AC}=\mathbf{c}$, so that diagonal $AD=\mathbf{b}+\mathbf{c}$ and $\vec{BC}=\vec{AC}-\vec{AB}=\mathbf{c}-\mathbf{b}$. Remember that $\mathbf{b}\cdot\mathbf{c}=bc\cos\theta$, *i. e.* module of $\mathbf{b}$ multiplied by the projection $\mathbf{c}$ on $\mathbf{b}$, etc. etc.

(b) *Prove that a parallelogram whose diagonals are equal is a rectangle.*

Refer Q. 5. $AD=BC$; $\therefore$ $\vec{AD}^2=\vec{BC}^2$ or $(\mathbf{b}+\mathbf{c})^2=(\mathbf{c}-\mathbf{b})^2$

or $4\mathbf{b}\cdot\mathbf{c}=0$, *i.e* $\vec{AB}\perp\vec{AC}$ and hence a rectangle.

Ex. 6. *If a straight line is equally inclined to three coplanar straight lines, prove that it is perpendicular to their plane.*

Taking O as origin, let OA, OB and OC be the three coplanar lines $\vec{OA}=\mathbf{a}$, $\vec{OB}=\mathbf{b}$, $\vec{OC}=\mathbf{c}$.

Again let $\overrightarrow{OD}=\mathbf{d}$ and we are given that OD is equally inclined to all the above lines.

$$\therefore \quad \cos\theta=\frac{\mathbf{a}\cdot\mathbf{d}}{ad}=\frac{\mathbf{b}\cdot\mathbf{d}}{bd}=\frac{\mathbf{c}\cdot\mathbf{d}}{cd}.$$

$$\therefore \quad \frac{\cos\theta}{\dfrac{\mathbf{a}}{a}}=\frac{\cos\theta}{\dfrac{\mathbf{b}}{b}}=\frac{\cos\theta}{\dfrac{\mathbf{c}}{c}}=\frac{\mathbf{d}}{d}.$$

But $\dfrac{\mathbf{a}}{a}\neq\dfrac{\mathbf{b}}{b}\neq\dfrac{\mathbf{c}}{c}$.

Hence the above will hold good only if $\cos\theta=0$ *i.e.* $\theta=90°$ which means that OD is perpendicular to all the three lines OA, OB, OC which in other words means that it is perpendicular to their plane.

Ex. 7. *If P be the middle point of the side BC of a triangle ABC, prove that $AB^2+AC^2=2\,(AP^2+BP^2)$.*

Choose P as origin and $\overrightarrow{PA}=\mathbf{a}$ and $\overrightarrow{PB}=\mathbf{b}$, so that $\overrightarrow{PC}=-\mathbf{b}$.

$\therefore \quad \overrightarrow{AB}=\overrightarrow{PB}-\overrightarrow{PA}=\mathbf{b}-\mathbf{a}$ and $\overrightarrow{AC}=\overrightarrow{PC}-\overrightarrow{PA}=-(\mathbf{b}+\mathbf{a})$ etc.

Square of a vector=square of its module. **(Property 7)**

Ex. 8. *Prove that in any triangle ABC,*

$$3\,(AB^2+BC^2+CA^2)=4\,(AP^2+BQ^2+CR^2)$$
$$=9\,(AG^2+BG^2+CG^2),$$

where P, Q, R are the middle points of the sides BC, CA and AB respectively of the triangle and G is the centroid.

Use $AB^2=\overrightarrow{AB}^2$. **(Property 7)**

Ex. 9. *In a qurdrilateral $ABCD$, prove that*

$$AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2,$$

where P and Q are the middle points of the diagonals AC and BD respectively.

Ex. 10. *Prove that the middle point of the hypotenuse of a right-angled triangle is equidistant from its vertices.* **(Pb. 60)**

Choose P the middle point of BC as origin and let the position vectors of A, B be $\mathbf{a}$ and $\mathbf{b}$; so that of C is $-\mathbf{b}$.

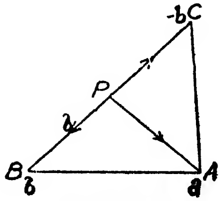


Fig No. 63

$\therefore$ $PB=PC$. We have to prove that $PA=PB=PC$

or $PA^2=PB^2$.

$$\vec{AB}^2+\vec{AC}^2=\vec{BC}^2 \quad \text{or} \quad (\mathbf{b}-\mathbf{a})^2+(-\mathbf{b}-\mathbf{a})^2=(-2\mathbf{b})^2$$

or $2\mathbf{b}^2+2\mathbf{a}^2=4\mathbf{b}^2$ or $\mathbf{b}^2=\mathbf{a}^2$ or $PB^2=PA^2=PC^2$.

Ex. 11. (a) *Prove that in any triangle the perpendiculars from the vertices upon the opposite sides are concurrent.*

(Agra 42, 47; Utkal 53; Luck. 54)

Also prove that the right bisectors of the sides are concurrent.

(Luck. 49)

Let the point of intersection of altitudes BQ and CR meet at O and taking this point as origin let the position vectors of vertices A, B and C be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

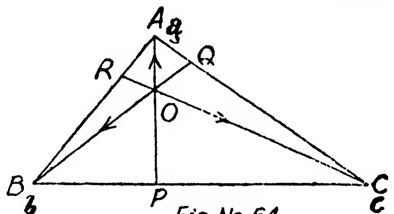


Fig. No. 64

Let AO produced meet BC at P. Then we should prove that AP is perp. to BC.

$$\vec{OB}=\mathbf{b}; \quad \therefore \quad \vec{BQ}=-\mu\mathbf{b}, \quad \vec{OC}=\mathbf{c}; \quad \therefore \quad \vec{CR}=-\nu\mathbf{c},$$

$$\vec{OA}=\mathbf{a}; \quad \therefore \quad \vec{AP}=-\lambda\mathbf{a}.$$

$$\vec{AC}=\mathbf{c}-\mathbf{a} \text{ and } \vec{AB}=\mathbf{b}-\mathbf{a}.$$

BQ is perp. to AC; $\therefore\ -\mu\mathbf{b}\cdot(\mathbf{c}-\mathbf{a})=0$; $\therefore\ \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}$,
CR is perp. to AB; $\therefore\ -\nu\mathbf{c}\cdot(\mathbf{b}-\mathbf{a})=0$; $\therefore\ \mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$.
$\therefore\ \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$ or $-\lambda\mathbf{a}\cdot(\mathbf{c}-\mathbf{b})=0$,
i.e. AP is perp. to BC.

Let the right bisector of sides BC and CA meet at O. Taking this point as origin let the position vectors of the vertices be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, so that the position vectors of the middle points D, E, F are $\frac{\mathbf{b}+\mathbf{c}}{2}, \frac{\mathbf{c}+\mathbf{a}}{2}, \frac{\mathbf{a}+\mathbf{b}}{2}$.

Now OD is perpendicular to BC.

$$\frac{\mathbf{b}+\mathbf{c}}{2}\cdot(\mathbf{c}-\mathbf{b})=0, \quad \therefore \quad \mathbf{b}^2=\mathbf{c}^2.$$

Similarly OE is perp. to CA; $\therefore\ \mathbf{c}^2=\mathbf{a}^2$.

Now we have to prove that OF is also perp. to AB which will be true if $\frac{\mathbf{a}+\mathbf{b}}{2}\cdot(\mathbf{b}-\mathbf{a})=0$, *i.e.* $\mathbf{b}^2=\mathbf{a}^2$ which is true.

$$\therefore \quad \mathbf{b}^2=\mathbf{c}^2=\mathbf{a}^2.$$

(b) *Prove that the median to the base of an isosceles triangle is perpendicular to the base.*

Ex. 12. (a) *Prove that the diagonals of a rhombus intersect at right angles.* **(Luck. 50)**

Taking A as origin, let the position vectors of B and D be $\mathbf{b}$ and $\mathbf{d}$, and since $AB=AD$, we have $\mathbf{b}^2=\mathbf{d}^2$. Also

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\mathbf{b}+\mathbf{d},$$

$$\overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}=\mathbf{d}-\mathbf{b},$$

$$\overrightarrow{AC}\cdot\overrightarrow{BD}=(\mathbf{d}+\mathbf{b})\cdot(\mathbf{d}-\mathbf{b})$$

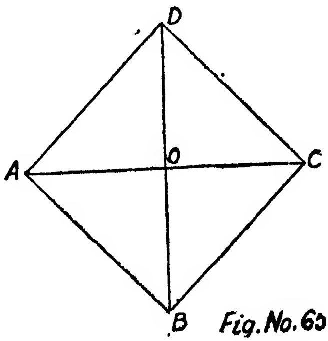


Fig. No. 65

$$=\mathbf{d}^2-\mathbf{b}^2 \text{ which is zero as } \mathbf{b}^2=\mathbf{d}^2.$$

Hence AC is perpendicular to BD.

(b) *If a point is equidistant from the vertex of a right-angled triangle, its join to the mid. point of the hypotenuse is perpendicular to the plane of the triangle.* **(Agra 52, 54)**

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of the right-angled triangle.

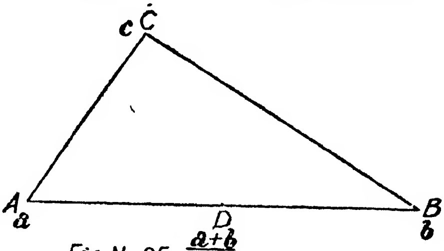


Fig. No. 95

$\therefore$ $AB^2=AC^2+BC^2$

or $(\mathbf{b}-\mathbf{a})^2=(\mathbf{c}-\mathbf{a})^2+(\mathbf{c}-\mathbf{b})^2$

or $$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}-\mathbf{c}^2 \qquad \ldots \qquad \ldots(1)$$

Let $\mathbf{p}$ be the position vector of any point P which is equidistant from the vertices.

$$\therefore \quad (\mathbf{p}-\mathbf{a})^2=(\mathbf{p}-\mathbf{b})^2=(\mathbf{p}-\mathbf{c})^2.$$

$$\therefore \quad 2\mathbf{p}\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2,\ 2\mathbf{p}\cdot(\mathbf{b}-\mathbf{c})=\mathbf{b}^2-\mathbf{c}^2$$

and $$2\mathbf{p}\cdot(\mathbf{c}-\mathbf{a})=\mathbf{c}^2-\mathbf{a}^2 \qquad \ldots \qquad \ldots(2)$$

or $$\left[\mathbf{p}-\frac{\mathbf{a}+\mathbf{b}}{2}\right]\cdot[\mathbf{a}-\mathbf{b}]=0$$

or DP is perpendicular to BA.

We should also prove that DP is perpendicular to AC for which we must have $\left[\mathbf{p}-\frac{\mathbf{a}+\mathbf{b}}{2}\right]\cdot(\mathbf{c}-\mathbf{a})$ equal to zero.

Now $$\mathbf{p}\cdot(\mathbf{c}-\mathbf{a})-\tfrac{1}{2}(\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}-\mathbf{a}^2-\mathbf{b}\cdot\mathbf{a})$$

or $$\frac{\mathbf{c}^2-\mathbf{a}^2}{2}-\tfrac{1}{2}(\mathbf{c}^2-\mathbf{a}^2)=0 \text{ from (1) and (2).}$$

DP is perpendicular to AB as well as AC and hence it is perpendicular to the plane ABC.

Ex. 13. *Prove that in a parallelopiped, the sum of the squares on the diagonals is equal to sum of the squares on the edges.*

The four diagonals are OP, CD, AE, BF, and clearly,

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})^2+(\mathbf{a}+\mathbf{b}-\mathbf{c})^2$$
$$+(\mathbf{b}+\mathbf{c}-\mathbf{a})^2+(\mathbf{c}+\mathbf{a}-\mathbf{b})^2$$
$$=4\,(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2)$$
$$=4\,(OA^2+OB^2+OC^2).$$

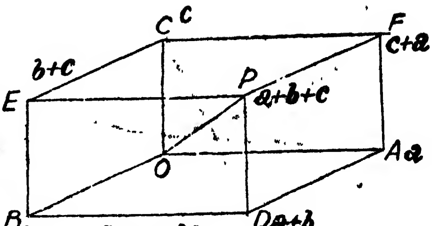


Fig. No. 66

$\because \quad \mathbf{a}^2=\overrightarrow{OA}^2=OA^2.$ **(Property 7)**

Ex. 14. *A line makes angles α, β, γ, δ with the diagonals of a cube ; prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$. Also prove that the angle between two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.*

Refer figure Q. 13 and let $\overrightarrow{OA}=\mathbf{i}$, $\overrightarrow{OB}=\mathbf{j}$ and $\overrightarrow{OC}=\mathbf{k}$, so that the diagonals are

$$\overrightarrow{OP}=\mathbf{i}+\mathbf{j}+\mathbf{k},\ \overrightarrow{CD}=(\mathbf{i}+\mathbf{j}-\mathbf{k}),\ \overrightarrow{AE}=\mathbf{j}+\mathbf{k}-\mathbf{i},\ \overrightarrow{BF}=\mathbf{k}+\mathbf{i}-\mathbf{j}.$$

Let OL be any line $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

$$\therefore\quad OL=\sqrt{(x^2+y^2+z^2)}.$$

$$\overrightarrow{OP}\,.\,\overrightarrow{OL}=OP\,.\,OL\cos\alpha,$$

$$(\mathbf{i}+\mathbf{j}+\mathbf{k})\,.\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\sqrt{(1+1+1)}\,.\,\sqrt{(x^2+y^2+z^2)}\cos\alpha$$

or $$\frac{x+y+z}{\sqrt{3}\sqrt{(x^2+y^2+z^2)}}=\cos\alpha.$$ **(Property 8)**

Similarly $\cos\beta=\dfrac{x+y-z}{\sqrt{3}\sqrt{(\Sigma x^2)}}$, $\cos\gamma=\dfrac{y+z-x}{\sqrt{3}\sqrt{(\Sigma x^2)}}$,

$$\cos\delta=\frac{z+x-y}{\sqrt{3}\sqrt{(\Sigma x^2)}}.$$

$$\therefore\quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$$
$$=\frac{(x+y+z)^2+(x+y-z)^2+(y+z-x)^2+(z+x-y)^2}{3\,(x^2+y^2+z^2)}=\frac{4}{3}.$$

Ex. 15. *Prove that sum of the squares of the distances of any point O from the angular points of a triangle exceeds the sum of the squares of its distances from the mid. points of its sides by sum of the squares of half the sides.* **(Agra 48)**

Ex. 16. *If O be the circum centre, G the centroid and H the ortho-centre of a triangle, prove that O, G, H are collinear and that G divides OH in the ratio 1 : 2.*

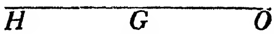


Fig. 67

Take O the circum-centre of the triangle ABC as the origin so that the points A, B, C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Also $OA=OB=OC$,

or $\qquad OA^2=OB^2=OC^2$, *i.e.* $\mathbf{a}^2=\mathbf{b}^2=\mathbf{c}^2$ ... ...(1)

Again position vector of centroid G is $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

From (1), $\mathbf{a}^2=\mathbf{b}^2$ or $\mathbf{b}^2=\mathbf{c}^2$ or $\mathbf{c}^2=\mathbf{a}^2$.

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=(\mathbf{b}+\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})=(\mathbf{c}+\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})=0$$

or $\qquad [(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{c}]\cdot(\mathbf{a}-\mathbf{b})=0,\ [(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{a}]\cdot[\mathbf{b}-\mathbf{c}]=0$

and $\qquad [(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{b}]\cdot[\mathbf{c}-\mathbf{a}]=0.$

Now if M be the point whose position vector is $\mathbf{a}+\mathbf{b}+\mathbf{c}$, then above relations show that

$$\overrightarrow{CM}\cdot\overrightarrow{BA}=0,\ \overrightarrow{AM}\cdot\overrightarrow{CB}=0,\ \overrightarrow{BM}\cdot\overrightarrow{AC}=0$$

i. e. $\qquad CM \perp BA$, $AM \perp CB$ and $BM \perp AC$.

Hence M is ortho-centre of the triangle which is given to be H.

$\therefore \quad \overrightarrow{OM}=\overrightarrow{OH}=\mathbf{a}+\mathbf{b}+\mathbf{c}=3\overrightarrow{OG}.$

$\therefore \quad O$, G and H are collinear.

$$\overrightarrow{OH}-\overrightarrow{OG}=2\overrightarrow{OG} \quad \text{or} \quad \overrightarrow{GH}=2\overrightarrow{OG}.$$

$\therefore \quad G$ divides OH in the ratio $1:2$.

Ex. 17. *If P be any point in the side AB of a triangle ABC, such that*

$$\lambda \cdot AP = \mu \cdot PB,$$

then prove that

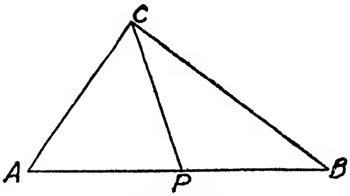


Fig. No. 68

$$\lambda \cdot CA^2 + \mu \cdot CB^2 = \lambda \cdot AP^2 + \mu \cdot BP^2 + (\lambda+\mu) \cdot CP^2.$$

$$CA^2 = \vec{CA}^2 = (\vec{CP} + \vec{PA})^2 = CP^2 + PA^2 + 2\vec{CP} \cdot \vec{PA} \ldots \qquad \ldots(1)$$

$$CB^2 = \vec{CB}^2 = (\vec{CP} + \vec{PB})^2 = CP^2 + PB^2 + 2\vec{CP} \cdot \vec{PB} \ldots \qquad \ldots(2)$$

Multiply (1) by λ and (2) by μ and add.

$$\lambda \cdot CA^2 + \mu \cdot CB^2 = (\lambda+\mu) \cdot CP^2 + \lambda \cdot AP^2 + \mu \cdot BP^2$$
$$+ 2\vec{CP} [\lambda \cdot \vec{PA} + \mu \cdot \vec{PB}]$$

Now $\lambda \cdot \vec{AP} = \mu \cdot \vec{PB}$ or $\mu \cdot \vec{PB} - \lambda \cdot \vec{AP} = 0$

or $\qquad \mu \cdot \vec{PB} + \lambda \cdot \vec{PA} = 0.$ Hence etc.

Ex. 18. *Prove by vectors that in any triangle ABC,*

$$a = b \cos C + c \cos B$$

and $\qquad a^2 = b^2 + c^2 - 2bc \cos A.$

(Agra 42)

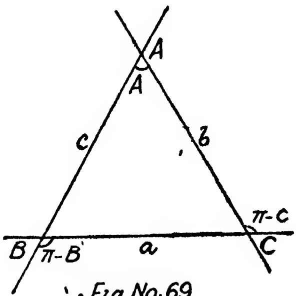


Fig. No. 69

Let $\vec{BC}, \vec{CA}$ and $\vec{AB}$ have modules a, b and c respectively.

Also $\vec{BC} + \vec{CA} + \vec{AB} = 0.$

$$\vec{BC} = -(\vec{CA} + \vec{AB}). \quad \text{Squaring,}$$

$$\vec{BC}^2 = \vec{CA}^2 + \vec{AB}^2 + 2\vec{CA} \cdot \vec{AB},$$

Squaring, we get

$$a^2=b^2+c^2+2bc\cos(\pi-A)$$
$$=b^2+c^2-2bc\cos A.$$

Again $\quad \vec{BC}\cdot\vec{BC}=-(\vec{CA}+\vec{AB})\cdot\vec{BC}$

or $\quad \vec{BC}^2=-\vec{CA}\cdot\vec{BC}-\vec{AB}\cdot\vec{BC}$

or $\quad a^2=-b\cdot a\cos(\pi-c)-c\cdot a\cos(\pi-B)$

or $\quad a=b\cos C+c\cos B$

Ex. 19. *Prove by vector method the following formula of plane trigonometry* :—

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

Let there be two unit vectors **i** and **j** along OX and OY, two perpendicular lines in the plane of the paper. If OP and OQ be any two lines in the same plane making angles α and β with OX respectively, then $\angle POQ=\alpha-\beta$.

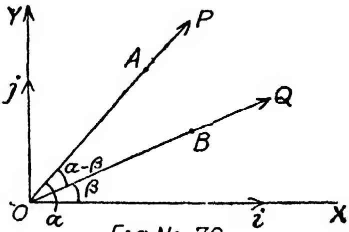


Fig No. 70

Again let $\vec{OA}$ and $\vec{OB}$ represent unit vectors along OP and OQ respectively so that their dot product is cosine of the angle between their directions,

i. e. $\quad \vec{OA}\cdot\vec{OB}=1.1.\cos(\alpha-\beta)=\cos(\alpha-\beta)\quad \ldots\quad \ldots\quad \ldots(1)$

But $\vec{OA}$ inclined at an angle α to the direction of **i** can be expressed as $\vec{OA}=\cos\alpha\cdot\mathbf{i}+\sin\alpha\cdot\mathbf{j}$

and similarly $\vec{OB}=\cos\beta\cdot\mathbf{i}+\sin\beta\cdot\mathbf{j}$.

$\therefore\quad \vec{OA}\cdot\vec{OB}=[\cos\alpha\cdot\mathbf{i}+\sin\alpha\cdot\mathbf{j}]\cdot[\cos\beta\cdot\mathbf{i}+\sin\beta\cdot\mathbf{j}]$

$\qquad =\cos\alpha\cos\beta+\sin\alpha\sin\beta\quad \ldots\quad \ldots\quad \ldots(2)$

$$\because \quad \mathbf{i}^2=\mathbf{j}^2=1 \quad \text{or} \quad \mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{i}=0.$$

Hence from (1) and (2), we get

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

Ex. 20. *Particles of masses $m_1, m_2, m_3, \ldots\ldots$ are placed at the points $A, B, C\ldots$ respectively and G is their C. M. Prove that for any point P,*

$$m_1AP^2+m_2BP^2+m_3CP^2\ldots\ldots=m_1AG^2+m_2BG^2+\ldots\ldots\ldots +(\Sigma\, m_1)\,PG^2.$$

It will be convenient if we choose the centre of mass G as origin and the position vectors of $A, B, C\ldots.$ be taken as $\mathbf{a}, \mathbf{b}, \mathbf{c}\ldots.$ respectively and that of P be $\mathbf{p}$.

$$\therefore \quad \overrightarrow{AP}=\overrightarrow{GP}-\overrightarrow{GA}=\mathbf{p}-\mathbf{a},\ \overrightarrow{BP}=\mathbf{p}-\mathbf{b} \text{ etc.}$$

Also since the c. m. is at the origin, we have $\Sigma\, m_1\mathbf{a}=0$ ...(1)

Also $AP^2=\overrightarrow{AP}^2=(\mathbf{p}-\mathbf{a})^2=\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{a}+\mathbf{a}^2.$

$$\begin{aligned}\therefore\quad & m_1AP^2+m_2BP^2+m_3CP^2+\ldots.\\ &=m_1(\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{a}+\mathbf{a}^2)+m_2(\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{b}+\mathbf{b}^2)+\ldots.\\ &=\mathbf{p}^2(m_1+m_2+m_3+\ldots)+2\mathbf{p}\cdot(\Sigma\, m_1\mathbf{a})\\ &\qquad\qquad +(m_1\mathbf{a}^2+m_2\mathbf{b}^2+\ldots.)\\ &=(\Sigma\, m_1)\,PG^2+m_1AG^2+m_2BG^2+\ldots.\\ &\qquad\qquad \because\ \Sigma\, m_1\mathbf{a}=0 \text{ by (2)}\end{aligned}$$

Ex. 21. *If there be four non-coplanar straight lines and unit vectors parallel to their directions be denoted by* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{d}$ *respectively and cos (a, b) stands for the angle between the lines which are parallel to* $\mathbf{a}$ *and* $\mathbf{b}$*, then prove that*

$$\begin{vmatrix} 1 & \cos(a,b) & \cos(a,c) & \cos(a,d)\\ \cos(b,a) & 1 & \cos(b,c) & \cos(b,d)\\ \cos(c,a) & \cos(c,b) & 1 & \cos(c,d)\\ \cos(d,a) & \cos(d,b) & \cos(d,c) & 1 \end{vmatrix}=0.$$

We know that there exists a linear relation between any four non-coplanar vectors and let it be

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+w\mathbf{d}=0 \quad \ldots \quad \ldots \quad \ldots(1)$$

Multiplying (1) scalarly by **a**, **b**, **c** and **d** in succession and noting that $\mathbf{a}\cdot\mathbf{a}=1$ as **a** is a unit vector and $\mathbf{a}\cdot\mathbf{b}=1\cdot1\cdot\cos(a, b)$, we get on multiplying (1) scalarly by **a**

$$x\cdot 1+y\cdot\cos(a, b)+z\cos(a, c)+w\cos(a, d)=0 \quad \ldots(2)$$

We can write down similar relations as above on multiplication of (1) scalarly by **b**, **c** and **d**. On eliminating x, y, z and w between the four relations thus obtained, we get the required result.

Ex. 22. *The position vectors of the foci of an ellipse are* **b** *and* $-\mathbf{b}$, *and the length of the major axis is 2a. Prove that the equation of the ellipse is*

$$a^4-a^2(\mathbf{r}+\mathbf{b}^2)+(\mathbf{b}\cdot\mathbf{r})^2=0.$$

We know that in the case of ellipse the sum of the focal distances of any point on it is equal to the length of major axis. Hence if **r** be any point on it, then

$$|\mathbf{r}+\mathbf{b}|+|\mathbf{r}-\mathbf{b}|=2a$$

or $$(\mathbf{r}+\mathbf{b})^2=[2a-|\mathbf{r}-\mathbf{b}|]^2.$$

[Square of a vector is square of its module]

or $$\mathbf{r}^2+2\mathbf{r}\cdot\mathbf{b}+\mathbf{b}^2=4a^2-4a\,|(\mathbf{r}-\mathbf{b})|+\mathbf{r}^2+\mathbf{b}^2-2\mathbf{r}\cdot\mathbf{b}$$

or $$[a^2-\mathbf{r}\cdot\mathbf{b}]=a\,|(\mathbf{r}-\mathbf{b}|\quad \text{Square again.}$$

$$a^4-2a^2\mathbf{r}\cdot\mathbf{b}+(\mathbf{r}\cdot\mathbf{b})^2=a^2(\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{b}+\mathbf{b}^2).$$

or $$a^4-a^2(\mathbf{r}^2+\mathbf{b}^2)+(\mathbf{r}\cdot\mathbf{b})^2=0.$$

Ex. 23. *Prove that the hyperbola whose foci are points* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *and whose transverse axis is 2a is given by*

$$|\mathbf{r}-\mathbf{b}_1|-|\mathbf{r}-\mathbf{b}_2|=2a.$$

Ex. 24. *Prove that*

$$\left(\frac{\mathbf{a}}{a^2}-\frac{\mathbf{b}}{b^2}\right)^2=\left(\frac{\mathbf{a}-\mathbf{b}}{ab}\right)^2.$$

§ 3. Vector Product. Definition.

(Agra 40, 51, 57, 58; Pb. 60; Raj. 57)

The vector (or cross) product of two vectors **a** *and* **b** *of moduli a and b respectively is a vector whose module is ab sin θ, θ being the angle between the directions of vectors* **a** *and* **b** *and whose direction is perpendicular to both* **a** *and* **b**, *this direction being regarded positive, if the rotation from* **a** *to* **b** *appears counter-clockwise. In case the rotation is from* **b** *to* **a**, *then it will be in clockwise direction and hence negative.*

Thus $\mathbf{a}\times\mathbf{b}=ab\sin\theta\,.\,\mathbf{n}$ where **n** is a unit vector perpendicular to both **a** and **b**.

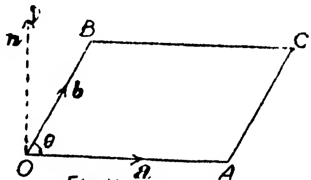


Fig. No. 71

1. Vector product is not commutative.

We have proved that dot product of two vectors is commutative, *i. e.* $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$, but this does not hold good in the case of cross product of two vectors **a** and **b**, *i. e.* $\mathbf{a}\times\mathbf{b}$ is not equal to $\mathbf{b}\times\mathbf{a}$ because the rotation which carries **a** to **b**, *i. e.* counter-clockwiseis opposite to that which carries **b** to **a**, *i. e.* clockwise. Of course the magnitudes of the two are same. but their sence is opposite.

Hence $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$.

Therefore we conclude that the vector product is not commutative, i. e. the factors in a vector product can only be interchanged if and only if the sign of the product is reversed.

2. Vector Product is associative.

Just as dot product of two vectors is associative, their cross product is also associative, *i, e.* if either factor **a** or **b** in the cross product is multiplied by a scalar *m*, then their product is also multiplied by that scalar,

i.e. $(m\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(m\mathbf{b})=m\,.\,(\mathbf{a}\times\mathbf{b})=m\,(ab\sin\theta\,\mathbf{n})$

Thus the vector product of two vectors is associative.

3. Cross product of two parallel vectors.

We know that $\mathbf{a}\times\mathbf{b}=ab\sin\theta\,.\,\mathbf{n}$.

In case **a** and **b** are parallel, then angle between them *i. e.* θ should be either 0 or 180° and in either case $\sin\theta=0$.

$\therefore$ $\mathbf{a}\times\mathbf{b}=0$. From here it also follows that $\mathbf{a}\times\mathbf{a}=0$.

Thus we conclude that vector product of two parallel or equal vectors is zero.

Conversely, if $\mathbf{a}\times\mathbf{b}=0$, $\therefore$ $ab\sin\theta\,\mathbf{n}=0$, then either $a=0$ or $b=0$, or $\sin\theta=0$, *i.e.* either of the vectors is a zero or null vector, and in case neither of the vectors is a zero vector, then $\sin\theta$ being zero shows that they are parallel.

Thus if cross product of two vectors neither of which is a zero vector vanishes, then these vectors are parallel.

4. Cross product of two perpendicular vectors.

In case the vectors are perpendicular, *i.e.* $\theta=90°$, then $\sin\theta=1$; $\therefore$ $\mathbf{a}\times\mathbf{b}=ab\,.\,\mathbf{n}$.

Thus the cross product of two perpendicular vectors is a vector whose module is equal to the product of the moduli of the given vectors and whose direction is such that **a, b** *and* **n** *form a right-handed system of mutually perpendicular vectors.*

5. Cross product of unit vectors.

In case **a** and **b** are unit vectors *i.e.* their moduli are each unity, then $\mathbf{a}\times\mathbf{b}=\sin\theta\,.\,\mathbf{n}$.

Thus the cross product of two unit vectors is a vector whose module is equal to the sine of the angle between the directions of the given vectors.

6. Unit vectors i, j, k. (Very important relations)

From above it is easy to deduce that

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$$

whereas $$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1$$

and $$\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i},$$

$$\mathbf{j}\times\mathbf{k}=\mathbf{i}=-\mathbf{k}\times\mathbf{j},$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k},$$
whereas $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{i}=0$, $\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{j}=0$, $\mathbf{k}\cdot\mathbf{i}=\mathbf{i}\cdot\mathbf{k}=0$.

7. We know that $\mathbf{a}\times\mathbf{b}=ab\sin\theta\,.\mathbf{n}$.

If $\mathbf{b}'$ be the component of $\mathbf{b}$ perpendicular to $\mathbf{a}$ then $\mathbf{b}'=b\sin\theta$. Also $\mathbf{a}$ and $\mathbf{b}'$ are in the same plane as $\mathbf{a}$ and $\mathbf{b}$ so that $\mathbf{n}$ remains the same.

$$\therefore\quad \mathbf{a}\times\mathbf{b}'=ab\sin\theta\,.\,\mathbf{n}=\mathbf{a}\times\mathbf{b}.$$

Thus we conclude that the vector product of two vectors remains unchanged if one factor is replaced by its normal component to the other.

8. The component of a vector r perpendicular to a given vector a. **(Delhi 59, Lucknow 52)**

Let a unit vector in the direction of $\mathbf{a}$ be denoted by $\mathbf{i}$ so that $\mathbf{a}=a\mathbf{i}$ and in a direction perpendicular to it be $\mathbf{j}$. Let the unit vector $\mathbf{k}$ be perpendicular to the plane $\mathbf{i}$ and $\mathbf{j}$. Now if $\mathbf{r}$ be any vector in the $\mathbf{i}\,\mathbf{j}$ plane inclined at an angle θ to $\mathbf{a}$ then its components in the directicn perpendicular to $\mathbf{a}$, *i.e.* $\mathbf{j}$ is

$$\mathbf{r}\sin\theta\,.\,\mathbf{j} \qquad \ldots \quad \ldots(1)$$

Now

$$\mathbf{a}\times\mathbf{r}=ar\sin\theta\,.\,\mathbf{k} \qquad \ldots \quad \ldots(2)$$

where $\mathbf{k}$ is a unit vector perpendicular to $\mathbf{i}\,\mathbf{j}$ plane in which both $\mathbf{a}$ and $\mathbf{r}$ lie. Now $\mathbf{k}\times\mathbf{i}=\mathbf{j}$.

$$\therefore\quad ar\sin\theta\,.\,\mathbf{j}=ar\sin\theta\,.\,\mathbf{k}\times\mathbf{i}=(\mathbf{a}\times\mathbf{r})\times\mathbf{i} \text{ by (2).}$$

$$\therefore\quad r\sin\theta\,.\,\mathbf{j}=\frac{(\mathbf{a}\times\mathbf{r})\times\mathbf{i}}{a}=\frac{(\mathbf{a}\times\mathbf{r})\times a\mathbf{i}}{a^2}$$

or component of $\mathbf{r}\perp$ to $\mathbf{a}=\dfrac{(\mathbf{a}\times\mathbf{r})\times\mathbf{a}}{a^2}=-\dfrac{\mathbf{a}\times(\mathbf{a}\times\mathbf{r})}{\mathbf{a}^2}$.

Note :—We have proved in § 2·11 P. 103 that the component of $\mathbf{r}$ in a direction perpendicular to $\mathbf{a}$ is

$$\mathbf{r}-\frac{\mathbf{r}\cdot\mathbf{a}}{a^2}\cdot\mathbf{a}.$$

We will show in the following pages that it is same as found above.

9. Distributive Law.

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

See § 5 Cor. 5 P. 138 for proof.

Thus in general

$$\begin{aligned}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\ldots.)\times(\mathbf{p}+\mathbf{q}+\mathbf{r}+\ldots.)\\ =\mathbf{a}\times\mathbf{p}+\mathbf{a}\times\mathbf{q}+\mathbf{a}\times\mathbf{r}+\ldots.\\ +\mathbf{b}\times\mathbf{p}+\mathbf{b}\times\mathbf{q}+\mathbf{b}\times\mathbf{r}+\ldots.\\ +\mathbf{c}\times\mathbf{p}+\mathbf{c}\times\mathbf{q}+\mathbf{c}\times\mathbf{r}+\ldots..\end{aligned}$$

10. Expression of vector product in terms of unit vectors.

If **a** and **b** be expressed in terms of unit vectors as

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}, \quad \therefore \quad a=\sqrt{(a_1^2+a_2^2+a_3^2)};$$

and

$$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}, \quad \therefore \quad b=\sqrt{(b_1^2+b_2^2+b_3^2)};$$

$$\therefore \quad \mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}).$$

$$ab\sin\theta\,.\,\mathbf{n}=(a_2b_3-a_3b_2)\,\mathbf{i}+(a_3b_1-a_1b_3)\,\mathbf{i}+(a_1b_2-a_2b_1)\,\mathbf{k}\ldots \qquad \ldots(1)$$

$\because$ $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$ and $\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}$ etc.

The above may be expressed in determinant form as

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \qquad \ldots \quad \ldots(2)$$

[Remember]

Squaring both sides of (1) and remembering that square of a vector is the dot product of a vector by itself and also that $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1$ and $\mathbf{i}\,.\,\mathbf{j}=\mathbf{j}\,.\,\mathbf{k}=\mathbf{k}\,.\,\mathbf{i}=0$,

$$a^2b^2\sin^2\theta=(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2.$$

$$\therefore \quad \sin^2\theta=\frac{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)\text{ or }a^2b^2}.$$

Again if l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of **a** and **b**, then

$$l_1=\frac{a_1}{\sqrt{(a_1^2+a_2^2+a_3^2)}} \text{ i.e. } \frac{a_1}{a},\ m_1=\frac{a_2}{a},\ n_1=\frac{a_3}{a}$$

and
$$l_2=\frac{b_1}{\sqrt{(b_1^2+b_2^2+b_3^2)}} \text{ i.e., } \frac{b_1}{b},\ m_2=\frac{b_2}{b},\ n_2=\frac{b_3}{b}.$$

$$\therefore\quad \sin^2\theta=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2.$$

In case the two vectors are parallel, then $\mathbf{a}\times\mathbf{b}=0$ and hence from determinant (2), we must have two rows identical,

i.e.,
$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}.$$

Again

$\sin^2\theta=1-\cos^2\theta$ and $l_1^2+m_1^2+n_1^2=1=l_2^2+m_2^2+n_2^2$ and putting the value of $\cos\theta$ from 10 of § 2 P. 104, we get the well known Lagrange's identity

$$(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2$$
$$=(l_1^2+m_1^2+n_1^2)\,(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2.$$

11. In case $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ ... ...(1)

Now if $\mathbf{b}=\mathbf{c}+k\mathbf{a}$, then

$$\mathbf{a}\times\mathbf{b}=\mathbf{a}\times(\mathbf{c}+k\mathbf{a})=\mathbf{a}\times\mathbf{c} \qquad \text{... ...(2)}$$
$$\because\quad \mathbf{a}\times\mathbf{a}=0.$$

From (1) and (2), we conclude that if $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$, then it does not mean that $\mathbf{b}=\mathbf{c}$ only but that $\mathbf{b}$ may differ from $\mathbf{c}$ by a vector which is parallel to $\mathbf{a}$ as $\mathbf{b}=\mathbf{c}+k\mathbf{a}$.

12. $(\mathbf{a}\times\mathbf{b})^2=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\cdot\mathbf{b})^2$.

$\mathbf{a}\times\mathbf{b}=ab\sin\theta\,.\,\mathbf{n}$, where $\mathbf{n}$ is a unit vector.

Squaring both sides, we get

$$(\mathbf{a}\times\mathbf{b})^2=a^2b^2\sin^2\theta,\quad \because\quad \mathbf{n}^2=1$$
$$=a^2b^2(1-\cos^2\theta)=a^2b^2-a^2b^2\cos^2\theta$$
$$=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\cdot\mathbf{b})^2,$$

$\because\quad a^2=\mathbf{a}^2$ and $b^2=\mathbf{b}^2$ and $\mathbf{a}\cdot\mathbf{b}=ab\cos\theta$.

13. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

Proceeding as in Q. 19 P. 115,

$$\overrightarrow{OA}=\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j},$$

$$\overrightarrow{OB}=\cos\beta\,\mathbf{i}+\sin\beta\,\mathbf{j}.$$

If OQ' makes an angle β with the direction of $\mathbf{i}$ in opposite direction, then

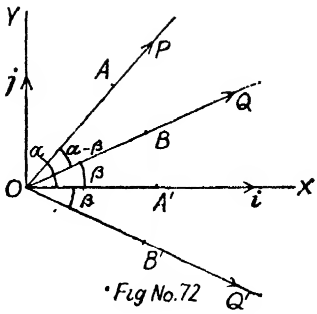


Fig No.72

$$\overrightarrow{OB'}=\cos\beta\,\mathbf{i}-\sin\beta\,\mathbf{j},\quad [\sin(-\beta)=-\sin\beta]$$

$$\overrightarrow{OB}\times\overrightarrow{OA}=(\cos\beta\,\mathbf{i}+\sin\beta\,\mathbf{j})\times(\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j})$$

or $\quad 1\cdot 1\cdot\sin(\alpha-\beta)\,\mathbf{n}=(\cos\beta\sin\alpha\,\mathbf{i}\times\mathbf{j}+\sin\beta\cos\alpha\,\mathbf{j}\times\mathbf{i}),$

where $\mathbf{n}$ is a unit vector $\perp$ to $\mathbf{i}$-$\mathbf{j}$ plane.

Now $\quad \mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=0$ and $\mathbf{i}\times\mathbf{j}=\mathbf{n}$ and $\mathbf{j}\times\mathbf{i}=-\mathbf{n}$;

$\therefore \quad \sin(\alpha-\beta)\,\mathbf{n}=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\,\mathbf{n}$

or $\quad \sin(\alpha-\beta)=(\sin\alpha\cos\beta-\cos\alpha\sin\beta).$

Similarly $\quad \overrightarrow{OB'}\times\overrightarrow{OA}=(\cos\beta\,\mathbf{i}-\sin\beta\,\mathbf{j})\,(\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j})$ and proceeding as above, we have

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

Exercise

Ex. 1. *Two vectors* **a** *and* **b** *are expressed in terms of unit vectors as follows :* $\mathbf{a}=3\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-2\mathbf{j}+4\mathbf{k}$. *What is the unit vector perpendicular to each of the vectors. Also determine the sine of the angle between the given vectors ?* **(Lucknow 48)**

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ 2 & -2 & 4\end{vmatrix}=8\,(\mathbf{i}-\mathbf{j}-\mathbf{k})\ \text{(from 10 P. 121).}$$

Now $\mathbf{a}\times\mathbf{b}$ represents a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and hence a unit vector in this direction is obtained by dividing $8(\mathbf{i}-\mathbf{j}-\mathbf{k})$ by its module, *i. e.* $\sqrt{(8^2+8^2+8^2)}$ or $8\sqrt{3}$.

Hence the required unit vector is

$$8\,\frac{(\mathbf{i}-\mathbf{j}-\mathbf{k})}{8\sqrt{3}}=\frac{\mathbf{i}-\mathbf{j}-\mathbf{k}}{\sqrt{3}}.$$

Again $(\mathbf{a}\times\mathbf{b})^2=8^2(\mathbf{i}-\mathbf{j}-\mathbf{k})^2=8^2\,.\,3.$

$(ab\sin\theta\,.\,\mathbf{n})^2=8^2\,.\,3$ but $a=\sqrt{(9+1+4)}=\sqrt{(14)}$

and $b=\sqrt{(4+4+16)}=\sqrt{(24)}$.

$$\therefore\quad a^2b^2\sin^2\theta\,.\,1=8^2\,.\,3\,;$$

$$\therefore\quad \sin^2\theta=\frac{8^2\,.\,3}{14\,.\,24}=\tfrac{4}{7}\,;\quad\therefore\quad \sin\theta=\frac{2}{\sqrt{7}}.$$

Ex. 2. *Prove that the unit vector perpendicular to each of the vectors* $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ *and* $3\mathbf{i}+4\mathbf{j}-\mathbf{k}$ *is* $\dfrac{-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}}{\sqrt{(155)}}$ *and the sine of the angle between them is* $\sqrt{\left(\dfrac{155}{156}\right)}$. **(Utkal 53)**

Ex. 3. *Taking* $\mathbf{a}$ *and* $\mathbf{b}$ *from Ex. 1 or 2, prove that* $\mathbf{a}\times\mathbf{b}$ *represents a vector which is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$.

Hint. You should show that dot product of both $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a}\times\mathbf{b}$ is zero.

Ex. 4. *Find the equation of the straight line through the point* $\mathbf{d}$ *and equally inclined to the vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *in the form* **(Imp.)**

$$\mathbf{r}=\mathbf{d}+s\left(\frac{\mathbf{a}}{a}+\frac{\mathbf{b}}{b}+\frac{\mathbf{c}}{c}\right)$$

and $$\mathbf{r}=\mathbf{d}+s\left[\frac{a\,(\mathbf{b}\times\mathbf{c})+b\,(\mathbf{c}\times\mathbf{a})+c\,(\mathbf{a}\times\mathbf{b})}{abc}\right].$$ **(Agra 60)**

Let the equation of th line through the point $\mathbf{d}$ be parallel to **unit vector** $\hat{\mathbf{k}}$ so that its equation is $\mathbf{r}=\mathbf{d}+t\hat{\mathbf{k}}$ ...(1)

Since the required line is equally inclined to vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, therefore they are equally inclined to $\hat{\mathbf{k}}$.

$$\therefore \quad \mathbf{a} \cdot \hat{\mathbf{k}} = a \cdot 1 \cos\theta,\ \mathbf{b} \cdot \hat{\mathbf{k}} = b \cdot 1 \cos\theta,\ \mathbf{c} \cdot \hat{\mathbf{k}} = c \cdot 1 \cos\theta.$$

$$\therefore \quad \frac{\mathbf{a}}{a} \cdot \hat{\mathbf{k}} = \frac{\mathbf{b}}{b} \cdot \hat{\mathbf{k}} = \frac{\mathbf{c}}{c} \cdot \hat{\mathbf{k}} = \cos\theta$$

or $$\hat{\mathbf{a}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{b}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{c}} \cdot \hat{\mathbf{k}} = \cos\theta \quad \ldots \quad \ldots (2)$$

Above shows that the resolved parts of $\hat{\mathbf{k}}$ in the directions of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are equal. In case their module be λ, then $\hat{\mathbf{k}}$ can be written as

$$\hat{\mathbf{k}} = \lambda\hat{\mathbf{a}} + \lambda\hat{\mathbf{b}} + \lambda\hat{\mathbf{c}} = \lambda\left(\frac{\mathbf{a}}{a} + \frac{\mathbf{b}}{b} + \frac{\mathbf{c}}{c}\right) \ldots \quad \ldots (3)$$

Substituting the value of $\hat{\mathbf{k}}$ from (3) in (1), we get the required equation,

as $$\mathbf{r} = \mathbf{b} + t\lambda\left(\frac{\mathbf{a}}{a} + \frac{\mathbf{b}}{b} + \frac{\mathbf{c}}{c}\right).$$ Replae $t\lambda$ by s.

Second form.

Again from (2), we get $(\hat{\mathbf{a}} - \hat{\mathbf{b}}) \cdot \hat{\mathbf{k}} = 0$,

$$(\hat{\mathbf{b}} - \hat{\mathbf{c}}) \cdot \hat{\mathbf{k}} = 0.$$

Now we know that dot product of two vectors is zero provided they are perpendicular. Hence we conclude that $\hat{\mathbf{k}}$ is perpendicular to both $(\hat{\mathbf{a}} - \hat{\mathbf{b}})$ and $(\hat{\mathbf{b}} - \hat{\mathbf{c}})$ and therefore $\hat{\mathbf{k}}$ is parallel to $(\hat{\mathbf{a}} - \hat{\mathbf{b}}) \times (\hat{\mathbf{b}} - \hat{\mathbf{c}})$ because $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\therefore \quad \hat{\mathbf{k}}=t\,(\hat{\mathbf{a}}\times\hat{\mathbf{b}}-\hat{\mathbf{a}}\times\hat{\mathbf{c}}-\hat{\mathbf{b}}\times\hat{\mathbf{b}}+\hat{\mathbf{b}}\times\hat{\mathbf{c}}).$$

Put $\hat{\mathbf{b}}\times\hat{\mathbf{b}}=0$ and $-\hat{\mathbf{a}}\times\hat{\mathbf{c}}=\hat{\mathbf{c}}\times\hat{\mathbf{a}}$

$$\therefore \quad \hat{\mathbf{k}}=t\,(\hat{\mathbf{a}}\times\hat{\mathbf{b}}+\hat{\mathbf{b}}\times\hat{\mathbf{c}}+\hat{\mathbf{c}}\times\hat{\mathbf{a}})$$

$$=t\left(\frac{\mathbf{a}\times\mathbf{b}}{ab}+\frac{\mathbf{b}\times\mathbf{c}}{bc}+\frac{\mathbf{c}\times\mathbf{a}}{ca}\right)$$

$$=t\left[\frac{a\,(\mathbf{b}\times\mathbf{c})+b\,(\mathbf{c}\times\mathbf{a})+c\,(\mathbf{a}\times\mathbf{b})}{abc}\right].$$

(b) *Prove that in a regular tetrahedron the perpendiculars from the vertices to the opposite faces meet at their centroids.*

Let the vector O be chosen as origin and the position vectors of the other vertices A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Since the tetrahedron is regular, therefore perpendicular from O on the face ABC will be equally inclined to OA, OB and OC *i. e.* vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$; as such its equation will be

$$\mathbf{r}=\lambda\left(\frac{\mathbf{a}}{a}+\frac{\mathbf{b}}{b}+\frac{\mathbf{c}}{c}\right).$$

But since the tetrahedron is regular,

$$\therefore \quad OA=OB=OC \quad \text{or} \quad a=b=c.$$

$$\therefore \quad \mathbf{r}=\frac{\lambda}{a}\,(\mathbf{a}+\mathbf{b}+\mathbf{c}) \qquad \ldots \quad \ldots \quad \ldots(1)$$

Equation to the plane ABC is

$$\mathbf{r}=(1-s-t)\,\mathbf{a}+s\mathbf{b}+t\mathbf{c} \qquad \ldots \quad \ldots \quad \ldots(2)$$

For the intersection of (1) and (2), on comparing the coefficients, we have

$$\frac{\lambda}{a}=1-s-t, \quad \frac{\lambda}{a}=s, \quad \frac{\lambda}{a}=t$$

$$\therefore \quad \frac{\lambda}{a}=1-\frac{\lambda}{a}-\frac{\lambda}{a} \quad \text{or} \quad \frac{\lambda}{a}=\tfrac{1}{3}=s=t.$$

Hence the position vector of the point where the

perpendicular from O meet ABC is

$$\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$$

which is the centroid of face ABC.

5. *By vector method establish the relation between the sides and angles of a triangle, i. e.* $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

Let $\vec{BC}, \vec{CA}, \vec{AB}$ be vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively.

$\therefore \quad \mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{o}.$

Now we know that cross product of two vectors one of which is zero and cross product of two equal vectors is zero.

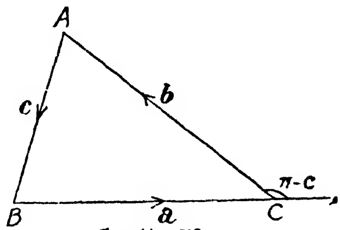


Fig No 73.

$$\therefore \quad \mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=0 \quad \text{or} \quad \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=0,$$

or $$\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a} \quad \because \quad -\mathbf{a}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}.$$

Similarly, $\mathbf{b}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=0$ or $\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}=0$

or $$\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}.$$

$$\therefore \quad |\mathbf{a}\times\mathbf{b}|=|\mathbf{b}\times\mathbf{c}|=|\mathbf{c}\times\mathbf{a}|$$

or $$ab\sin(\pi-C)=bc\sin(\pi-A)=ca\sin(\pi-B)$$

or $$ab\sin C=bc\sin A=ca\sin B.$$

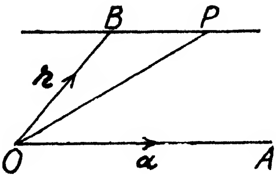
Dividing throughout by abc, we get

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

(b) *If the vector product of a constant vector* $\vec{OA}$ *with a variable vector* $\vec{OB}$ *in a fixed plane AOB be a constant vector, show that locus of B is a straight line parallel to OA.*

(Luck. B. Sc. 46, 49, 55)

$\overrightarrow{OA}=\mathbf{a}$ say and $\overrightarrow{OB}=\mathbf{r}$.



Now $\mathbf{a}\times\mathbf{r}$=constant given.

$$\mathbf{a}\times\mathbf{r}=k=\mathbf{a}\times\overrightarrow{OP}$$

$$=\mathbf{a}\times\mathbf{b} \text{ where } \overrightarrow{OP}=\mathbf{b} \text{ say.}$$

$$\therefore \quad \mathbf{a}\times(\mathbf{r}-\mathbf{b})=0.$$

Since the cross product is zero, therefore $\mathbf{r}-\mathbf{b}$ is parallel to $\mathbf{a}$.

$$\therefore \quad \mathbf{r}-\mathbf{b}=t\mathbf{a} \quad \text{or} \quad r=\mathbf{b}+t\mathbf{a}$$

Above represents a line through P parallel to OA. Hence the locus of B is as given.

Ex. 6. *Find the vector area of a triangle OAB where $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and they are inclined at an angle θ and hence find the vector area of a triangle whose vertices are the points* **a**, **b** *and* **c**.

(Utkal 50)

We know that $\mathbf{a}\times\mathbf{b}=ab\sin\theta\,.\,\mathbf{n}$.

Now area of $\triangle OAB=\frac{1}{2}OA\,.\,OB\sin\theta=\frac{1}{2}ab\sin\theta$.

$$\therefore \quad \mathbf{a}\times\mathbf{b}=ab\sin\theta\,.\,\mathbf{n}=2\triangle\mathbf{n}.$$

Hence vector area of $\triangle OAB=\frac{1}{2}\mathbf{a}\times\mathbf{b}$.

Also the vector area of the parallelogram two adjacent sides of which are **a** and **b** is $\mathbf{a}\times\mathbf{b}$.

Now if the position vectors of A, B and C be **a**, **b** and **c**, then $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$ and $\overrightarrow{BA}=\mathbf{a}-\mathbf{b}$.

Therefore vector area of $\triangle ABC$ is

$$\frac{1}{2}(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{b})$$

$$=\frac{1}{2}(\mathbf{c}\times\mathbf{a}-\mathbf{c}\times\mathbf{b}-\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b})$$

$$=\frac{1}{2}(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}) \qquad \textbf{[Remember]}$$

$$\because \quad \mathbf{b}\times\mathbf{b}=0 \text{ and } -\mathbf{b}\times\mathbf{a}=\mathbf{a}\times\mathbf{b}.$$

Note. In case the three points are collinear, then clearly $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=0$.

Ex. 7. *In a triangle ABC points D, E, F are taken on the sides BC, CA and AB respectively such that*

$$BD : DC=CE : EA=AF : FB=n : 1,$$

prove that $$\triangle DEF=\frac{n^2-n+1}{(n+1)^2}\triangle ABC.$$

Taking A as origin let the position vectors of B and C be $\mathbf{b}$ and $\mathbf{c}$ respectively. Therefore the position vectors of F, D and E are respectively

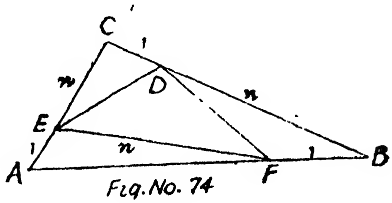


Fig. No. 74

$$F\ \frac{n\mathbf{b}}{n+1},\ D\ \frac{n\mathbf{c}+\mathbf{b}}{n+1},\ E\ \frac{\mathbf{c}}{n+1}.$$ [§ 1 P. 36]

$$\therefore\quad \overrightarrow{EF}=\overrightarrow{AF}-\overrightarrow{AE}=\frac{n\mathbf{b}-\mathbf{c}}{n+1}$$

$$\overrightarrow{ED}=\overrightarrow{AD}-\overrightarrow{AE}=\frac{n\mathbf{c}+\mathbf{b}-n\mathbf{b}}{n+1}.$$

Vector area of $\triangle ABC=\frac{1}{2}(\mathbf{b}\times\mathbf{c})$. [by Q. 6.]

Vector area of $\triangle EFD=\frac{1}{2}(\overrightarrow{EF}\times\overrightarrow{ED})$

$$=\frac{1}{2}\cdot\frac{1}{(n+1)^2}(n\mathbf{b}-\mathbf{c})\times[n\mathbf{c}+(1-n)\,\mathbf{b}]$$

$$=\frac{1}{2}\frac{1}{(n+1)^2}[n^2\mathbf{b}\times\mathbf{c}-(1-n)\,\mathbf{c}\times\mathbf{b}]$$

$\because\ \mathbf{b}\times\mathbf{b}=\mathbf{c}\times\mathbf{c}=0$

$$=\frac{1}{2}\frac{1}{(n+1)^2}(n^2-n+1)\,(\mathbf{b}\times\mathbf{c}),\quad \because\ \mathbf{c}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{c})$$

$$=\frac{n^2-n+1}{(n+1)^2}\triangle ABC.\quad \because\ \tfrac{1}{2}(\mathbf{b}\times\mathbf{c})=\triangle ABC$$

Note—In case D, E, F are the mid. points, then putting $n=1$, we get $\Delta EFD=\Delta DEF=\frac{1}{4}\Delta ABC$.

Ex. 8. *The internal bisectors of the angles of a triangle ABC meet the opposite sides in D, E, F; show that*

$$\Delta DEF=\frac{2abc}{(a+b)\,(b+c)\,(c+a)}\,\Delta ABC,$$

a, b, c being the lengths of BC, CA and AB respectively.

Refer fig. Ex. 7.

We know that internal bisector of an angle divides the opposite side in the ratio of the arms of the angle *i.e.* F divides AB in the ratio $CA : CB$, *i.e.* $b : a$ etc.

$\therefore$ position vectors of F, D and E are respectively

$$F\ \frac{b\mathbf{b}+\mathbf{a}\,.\,0}{b+a},\quad D\ \frac{c\mathbf{c}+b\mathbf{b}}{c+b},\quad E\ \frac{\mathbf{a}\,.\,0+c\mathbf{c}}{a+c},$$

$$\therefore\quad \overrightarrow{EF}=\overrightarrow{AF}-\overrightarrow{AE}=\frac{b\mathbf{b}}{b+a}-\frac{c\mathbf{c}}{a+c}=\frac{(a+c)\,b\mathbf{b}-(b+a)\,c\mathbf{c}}{(a+b)\,(c+a)},$$

$$\overrightarrow{ED}=\overrightarrow{AD}-\overrightarrow{AE}=\frac{c\mathbf{c}+b\mathbf{b}}{c+b}-\frac{c\mathbf{c}}{c+a}$$

$$=\frac{c\,(a-b)\,\mathbf{c}+b\,(c+a)\,\mathbf{b}}{(c+b)\,(c+a)}.$$

$$\therefore\ \Delta EFD=\tfrac{1}{2}\overrightarrow{EF}\times\overrightarrow{ED}.$$

$$=\tfrac{1}{2}\left[\frac{(a+c)\,b\mathbf{b}-(b+a)\,c\mathbf{c}}{(a+b)\,(c+a)}\right]\times\left[\frac{c\,(a-b)\,\mathbf{c}+b\,(c+a)\,\mathbf{b}}{(c+b)\,(c+a)}\right].$$

Now multiply 1st bracket above and below by $(b+c)$ and 2nd by $(a+b)$.

$$=\frac{1}{2\,(a+b)^2\,(b+c)^2\,(c+a)^2}\,[\{(a+c)\,(b+c)\,b\mathbf{b}-(b+a)\,(b+c)\,c\mathbf{c}\}$$

$$\times\{c\,(a-b)\,(a+b)\,\mathbf{c}+b\,(c+a)\,(a+b)\,\mathbf{b}\}].$$

Now keeping in view that $\mathbf{b}\times\mathbf{b}=\mathbf{c}\times\mathbf{c}=0$ and $-\mathbf{c}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$, we get

$$\Delta EFD = \frac{(\mathbf{b}\times\mathbf{c})}{2\,(a+b)^2\,(b+c)^2\,(c+a)^2}\,[(a+c)\,(b+c)\,b\,.\,c\,(a-b)\,(a+b)$$
$$+(b+a)\,(b+c)\,c\,.\,b\,(c+a)\,(a+b)]$$
$$=\frac{\Delta\ ABC}{(a+b)^2\,(b+c)^2\,(c+a)^2}.bc\,(a+b)\,(b+c)\,(c+a)\,[a-b+a+b$$
$$=\frac{2abc}{(a+b)\,(b+c)\,(c+a)}\,\Delta\ ABC.$$

Ex. 9. *Prove that* $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2\mathbf{a}\times\mathbf{b}$ *and interpret it.*
(Agra 47, Lucknow 56, Andhra 36)

We know that

$\mathbf{a}\times\mathbf{a}=\mathbf{b}\times\mathbf{b}=0$

and $-\mathbf{b}\times\mathbf{a}=\mathbf{a}\times\mathbf{b}$.

$\therefore$ $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})$

$=2\mathbf{a}\times\mathbf{b}$.

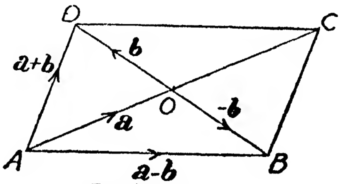


Fig. No. 75

Interpretation :—

In the parallelogram $ABCD$ let $\overrightarrow{AO}=\mathbf{a}$ and $\overrightarrow{OD}=\mathbf{b}$.

$\therefore$ $\overrightarrow{OB}=-\mathbf{b}$ and hence $\overrightarrow{AB}=\mathbf{a}-\mathbf{b}$ and $\overrightarrow{AD}=\mathbf{a}+\mathbf{b}$.

$\therefore$ $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})$ represents the area of parallelogram and it being $2\mathbf{a}\times\mathbf{b}$, *i. e.* twice the area of the parallelogram whose adjacent sides are semi-diagonals of the first parallelogram.

Ex. 10. *Prove that the area of the parallelogram determined by the vectors* $\mathbf{a}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ *and* $\mathbf{b}=-3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ *is* $6\sqrt{5}$.

Ex. 11. *Given the vector* $\mathbf{r}=a\mathbf{i}+b\mathbf{j}$ *where* $\mathbf{i}$, $\mathbf{j}$ *are unit vectors in the direction of the axes, find an expression for the vector of the same length perpendicular to the given vector through the origin.* **(Lucknow 46)**

The module of the given vector is $\sqrt{(a^2+b^2)}$ and it lies in $\mathbf{i}, \mathbf{j}$ plane and therefore $\mathbf{i}\times\mathbf{j}$ represents a vector perpendicular to $\mathbf{i}$ and $\mathbf{j}$ plane and hence perpendicular

to the given vector which lies in this plane. Since its length is to be the same as that of the given vector, hence it should be $\sqrt{(a^2+b^2)}\ \mathbf{i}\times\mathbf{j}$.

Ex. 12. *Prove that the area of the triangle formed by joining the middle point of one of the non-parallel sides of a trapezium to the extremities of the opposite side is half that of the trapezium.*

(Agra 45, 57)

Let P the middle point of the oblique side BC be joined to the extremities of the other side OA, then we have to prove that

$$\triangle OAP=\tfrac{1}{2}OABC.$$

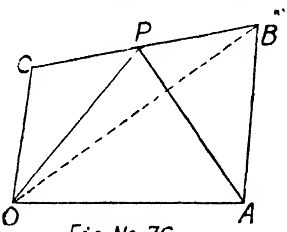


Fig. No 76

Let $\overrightarrow{OA}$ be $\mathbf{a}$ and $\overrightarrow{AB}$ be $\mathbf{b}$

so that position vector of B, *i. e.* $\overrightarrow{OB}=\mathbf{a}+\mathbf{b}$.

Now OC being parallel to $\overrightarrow{AB}$, $\therefore$ $\overrightarrow{OC}=k\overrightarrow{AB}=k\mathbf{b}$.

The position vectors of B and C are $\mathbf{a}+\mathbf{b}$ and $k\mathbf{b}$; therefore the position vector of P the middle point of BC is

$$\tfrac{1}{2}(\mathbf{a}+\mathbf{b}+k\mathbf{b}).$$

$$\triangle OAP=\tfrac{1}{2}\overrightarrow{OA}\times\overrightarrow{OP}=\tfrac{1}{2}\mathbf{a}\times\tfrac{1}{2}(\mathbf{a}+\mathbf{b}+k\mathbf{b})$$
$$=\tfrac{1}{4}(\mathbf{a}\times\mathbf{b}+k\mathbf{a}\times\mathbf{b})=\tfrac{1}{4}(k+1)\,\mathbf{a}\times\mathbf{b} \quad \ldots(1)$$

$$\overrightarrow{\triangle OAB}=\tfrac{1}{2}\overrightarrow{OA}\times\overrightarrow{OB}=\tfrac{1}{2}\mathbf{a}\times(\mathbf{a}+\mathbf{b})=\tfrac{1}{2}\mathbf{a}\times\mathbf{b},$$

$$\overrightarrow{\triangle OBC}=\tfrac{1}{2}\overrightarrow{OB}\times\overrightarrow{OC}=\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\times k\mathbf{b}=\tfrac{1}{2}k\mathbf{a}\times\mathbf{b};$$

$$\therefore \quad \text{trapezium } \overrightarrow{OABC}=\overrightarrow{\triangle OAB}+\overrightarrow{\triangle OBC}=\tfrac{1}{2}(k+1)\,\mathbf{a}\times\mathbf{b}.$$

$$\therefore \quad 2\overrightarrow{\triangle OAP}=\text{trapezium } \overrightarrow{OABC}.\ \text{from (1)}.$$

Ex. 13. *Q and R are the middle points of the sides AC and AB of a triangle ABC respectively. CP is drawn parallel to AB and meets BQ produced in P. Prove that $\triangle RQP = \triangle RCQ$ and each equal to one-fourth of $\triangle ABC$.*

Taking A as origin let the points B and C be $\mathbf{b}$ and $\mathbf{c}$ respectively so that R and Q are $\frac{1}{2}\mathbf{b}$ and $\frac{1}{2}\mathbf{c}$ repectively.

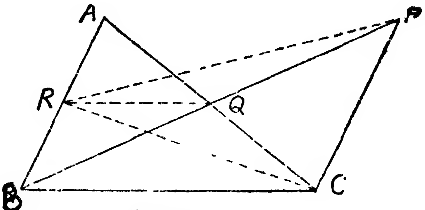


Fig No 77

$$\therefore \quad \triangle ABC = \tfrac{1}{2}\overrightarrow{BC} \times \overrightarrow{BA} = \tfrac{1}{2}(\mathbf{c}-\mathbf{b}) \times (-\mathbf{b})$$
$$= -\tfrac{1}{2}\mathbf{c} \times \mathbf{b} = \tfrac{1}{2}\mathbf{b} \times \mathbf{c}.$$

Equation to $\overrightarrow{CP}$ is $\mathbf{r} = \mathbf{c} + t\mathbf{b}$ as it is parallel to AB.

Equation to $\overrightarrow{BP}$ is $\mathbf{r} = \mathbf{b} + s(\mathbf{c}/2 - \mathbf{b})$ as it passes through Q *i. e.* $\mathbf{c}/2$ and B *i. e.* $\mathbf{b}$.

These lines intersect at P and hence on comparing, we get

$$1 = s/2 \text{ and } t = 1 - s\,; \quad \therefore \quad s = 2 \text{ and } t = -1.$$

Therefore the position vector of P is $\mathbf{c} - \mathbf{b}$.

$$\triangle RQP = \tfrac{1}{2}\overrightarrow{RQ} \times \overrightarrow{RP} = \tfrac{1}{2}(\tfrac{1}{2}\mathbf{c} - \tfrac{1}{2}\mathbf{b}) \times (\mathbf{c} - \mathbf{b} - \tfrac{1}{2}\mathbf{b})$$
$$= \tfrac{1}{2}(-\tfrac{3}{4}\mathbf{c} \times \mathbf{b} - \tfrac{1}{2}\mathbf{b} \times \mathbf{c}) = \tfrac{1}{2}(\tfrac{3}{4}\mathbf{b} \times \mathbf{c} - \tfrac{1}{2}\mathbf{b} \times \mathbf{c})$$
$$= \tfrac{1}{8}\mathbf{b} \times \mathbf{c} = \tfrac{1}{4}.\tfrac{1}{2}\mathbf{b} \times \mathbf{c} = \tfrac{1}{4}\triangle ABC.$$

Similarly $\triangle RCQ = \frac{1}{2}(\overrightarrow{RC} \times \overrightarrow{RQ})$ etc.

$$= \tfrac{1}{8}(\mathbf{b} \times \mathbf{c}) = \tfrac{1}{4}\triangle \overrightarrow{ABC}.$$

Ex. 14. *AC and BD are two diagonals of a quadrilateral. Prove that its area is $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$.*

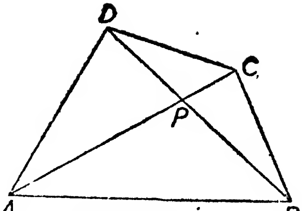


Fig No. 78

If P be the intersection of the diagonals, then quadrilateral is

$$\Delta \overrightarrow{PAB}+\Delta \overrightarrow{PBC}+\Delta \overrightarrow{PCD}+\Delta \overrightarrow{PDA}$$

$$=\tfrac{1}{2}(\overrightarrow{PA}\times\overrightarrow{PB}+\overrightarrow{PB}\times\overrightarrow{PC}+\overrightarrow{PC}\times\overrightarrow{PD}+\overrightarrow{PD}\times\overrightarrow{PA})$$

$$=\tfrac{1}{2}\{(-\overrightarrow{PB}\times\overrightarrow{PA}+\overrightarrow{PD}\times\overrightarrow{PA})+(\overrightarrow{PB}\times\overrightarrow{PC}-\overrightarrow{PD}\times\overrightarrow{PC})\}$$

$$=\tfrac{1}{2}[(\overrightarrow{BP}+\overrightarrow{PD})\times\overrightarrow{PA}+(-\overrightarrow{BP}-\overrightarrow{PD})\times\overrightarrow{PC}]$$

$$=\tfrac{1}{2}(\overrightarrow{BP}+\overrightarrow{PD})\times(\overrightarrow{PA}-\overrightarrow{PC})=\tfrac{1}{2}\overrightarrow{BD}\times(\overrightarrow{PA}+\overrightarrow{CP})=\tfrac{1}{2}\overrightarrow{BD}\times\overrightarrow{CA}$$

$$=-\tfrac{1}{2}\overrightarrow{BD}\times\overrightarrow{AC}=\tfrac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}.$$

15. *A line EF is drawn parallel to base BC of a triangle ABC meeting AC and AB in E and F respectively. If BR and CQ be drawn parallel to AC, AB respectively to meet EF in R and Q respectively, then prove that* $\Delta ARB=\Delta ACQ$. **(Agra 59)**

§ 4. Product of three vectors.

Scalar triple product and vector triple product.

(Agra 37, 40, Raj. 56, 57)

We have already seen that the dot product of two vectors **a** and **b** is a mere sclaar whereas the cross product of two vectors **a** and **b** is a vector itself,

i. e. $\mathbf{a}\cdot\mathbf{b}=ab\cos\theta$ (scalar)

and $\mathbf{a}\times\mathbf{b}=ab\sin\theta\,.\,\mathbf{n}$, (vector)

where **n** is a unit vector perpendicular to the plane of given vectors.

Now since $\mathbf{a}\times\mathbf{b}$ is a vector, we can multiply it both

scalarly and vectorially by another vector say **c**. **The former will be called scalar triple product and the latter vector triple product.**

$\therefore$ $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ or $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ is scalar triple product of the vector **a**, **b** and **c** whereas $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=-(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is vector triple product of the vectors **a**, **b** and **c**.

Naturally the questions arises : What do $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ and $\mathbf{c}\times(\mathbf{a}\cdot\mathbf{b})$ stand for ?

Since $\mathbf{a}\cdot\mathbf{b}$ is a scalar and as such $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ may be written as $\mathbf{a}\cdot\mathbf{b}\ \mathbf{c}$, it simply represents a vector in the direction of **c** whose module is $\mathbf{a}\cdot\mathbf{b}$ times that of **c**. **Thus the dot product of two vectors can occur as a coefficient of a third vector.**

Again as above $\mathbf{a}\cdot\mathbf{b}$ being a scalar and not a vector, therefore $\mathbf{c}\times(\mathbf{a}\cdot\mathbf{b})$ is meaningless.

§ 5. **Geometrical interpretation of scalar triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.** **(Agra 59)**

In the adjoining parallelopiped the three coterminous edges *OA*, *OB* and *OC* represent in magnitude and direction the vectors **a**, **b** and **c** respectively.

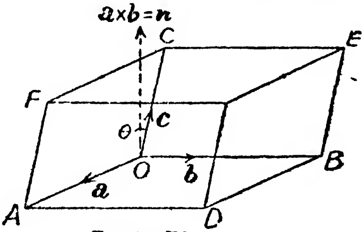


Fig. No 79

Therefore $\mathbf{a}\times\mathbf{b}$ represents a vector **n** whose magnitude is the area of the parallelogram *OADB* whose direction is perpendicular to the plane of the face *OADB*. Now if θ be the angle between the directions of $\mathbf{a}\times\mathbf{b}$ *i. e.* **n** and that of **c** then $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ *i. e.* $\mathbf{n}\cdot\mathbf{c}=$area $OADB\cdot c\cos\theta$ which we know represents the volume of the parallelopiped. The value of the scalar triple product is +ive when θ is acute *i. e.* **a**, **b**, **c** form a right-handed system of vectors.

In a similar manner we can show that $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$ and $(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$ also represents the volume of the above parallelopiped.

Also we know that $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ and as such

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \quad \ldots \quad \ldots \quad \ldots(1)$$

$$(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \quad \ldots \quad \ldots \quad \ldots(2)$$

$$(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \quad \ldots \quad \ldots \quad \ldots(3)$$

and each equal to volume.

Now we find that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ from (1) and (2).

We find that we have interchanged the position of dot and cross in above scalar triple product, but the cyclic order of the factors is maintained.

Similarly,

$$(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \text{ and } (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}).$$

Thus we conclude that in the scalar triple product the position of dot and cross can be interchanged at pleasure provided we maintain the cyclic order of the vectors a, b and c. (Pb. 60, Agra 59)

Effect of changing the cyclic order :—

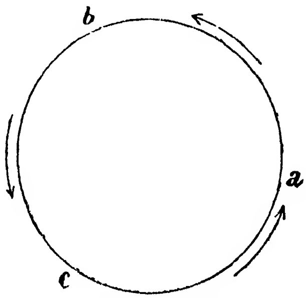


Fig No. 80

Now we know that

$(\mathbf{a}\times\mathbf{b})=-(\mathbf{b}\times\mathbf{a})$.

$$\therefore \quad V=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=-(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c}$$

$$=-\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}),$$

$$(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=-(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a}$$

$$=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}),$$

$$(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}=-(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b}$$

$$=-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}).$$

Thus we observe that by changing the cyclic order of the vector **a, b, c** or **b, c, a** or **c, a, b** to **b, a, c** or **a, c, b** or **c, b, a** the sign of the scalar triple product is changed.

Also we see that $-(\mathbf{a}\times\mathbf{c})\,.\,\mathbf{b}=-\mathbf{a}\,.\,(\mathbf{c}\times\mathbf{b})$ *and so on, i.e. the position of dot and cross can be changed at pleasure whether you maintain the cyclic order of* **a**, **b** *and* **c** *or not ; if, however, you change the cyclic order, the sign should be changed.*

Notation :—The scalar triple product of three vectors is generally written as $[\mathbf{abc}]$. Thus

$$[\mathbf{abc}]=[\mathbf{bca}]=[\mathbf{cab}]=-[\mathbf{acb}]=-[\mathbf{cba}]=-[\mathbf{bac}]$$

In each of the above different forms of scalar triple product the position of dot and cross can be changed.

$$\therefore\quad [\mathbf{i\,j\,k}]=\mathbf{i}\,.\,(\mathbf{j}\times\mathbf{k})=\mathbf{i}\,.\,\mathbf{i}=1,\quad \because\quad \mathbf{j}\times\mathbf{k}=\mathbf{i}.$$

Cor. 1. Condition for three vectors to be coplanar.

$[\mathbf{abc}]=\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c})$ where **a**, **b**, **c** are three coplanar vectors. Now $\mathbf{b}\times\mathbf{c}$ represents a vector which is perpendicular to the plane containing **b** and **c** in which also lies the vector **a** and hence $\mathbf{b}\times\mathbf{c}$ is perpendicular to **a**. Therefore $\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c})=0$ being the dot product of two perpendicular vectors **a** and $\mathbf{b}\times\mathbf{c}$. Thus $[\mathbf{abc}]=0$ when the three vectors are coplanar.

Converse :—If $[\mathbf{abc}]=0$, *i.e.* $\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c})=0$ showing that $\mathbf{b}\times\mathbf{c}$ is perpendicular to **a**. But $\mathbf{b}\times\mathbf{c}$ is perpendicular to the plane containing **b** and **c** and hence **a** should also lie in the plane of **b** and **c**, *i.e.* **a**, **b**, **c** should be coplanar.

Cor. 2. Scalar triple product when two of the vectors are equal.

$[\mathbf{aac}]=\mathbf{a}\,.\,(\mathbf{a}\times\mathbf{c})$. Now $\mathbf{a}\times\mathbf{c}$ is perpendicular to the plane containing **a** and **c** and therefore $\mathbf{a}\,.\,(\mathbf{a}\times\mathbf{c})=0$ or otherwise also $[\mathbf{aac}]=\mathbf{a}\,.\,(\mathbf{a}\times\mathbf{c})=(\mathbf{a}\times\mathbf{a})\,.\,\mathbf{c}$ because the position of dot and cross can be changed.

$$\therefore\quad [\mathbf{aac}]=(\mathbf{a}\times\mathbf{a})\,.\,\mathbf{c}=0,\qquad \because\quad (\mathbf{a}\times\mathbf{a})=0.$$

Cor. 3. Scalar triple product when two of the vectors are parallel.

Let **a** and **b** be parallel so that $\mathbf{b}=k\,\mathbf{a}$ where k is a scalar.

$$\therefore \quad [\mathbf{abc}]=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{a}\cdot(k\mathbf{a}\times\mathbf{c})=k\,.\,\mathbf{a}\cdot(\mathbf{a}\times\mathbf{c}).$$

$$\therefore \quad k\,[\mathbf{aac}]=0. \qquad \textbf{[by Cor. 2]}.$$

Hence from Cor. 1, 2 and 3, we have the following :

The scalar triple product of three vectors will be zero when they are coplanar or two of them are equal or parallel.

Cor. 4. We know that any vector can be expressed in terms of three non-coplanar vectors as

$$\mathbf{a}=a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n},$$

$$\mathbf{b}=b_1\mathbf{l}+b_2\mathbf{m}+b_3\mathbf{n},$$

$$\mathbf{c}=c_1\mathbf{l}+c_2\mathbf{m}+c_3\mathbf{n};$$

$$\therefore \quad \mathbf{b}\times\mathbf{c}=(b_1\mathbf{l}+b_2\mathbf{m}+b_3\mathbf{n})\times(c_1\mathbf{l}+c_2\mathbf{m}+c_3\mathbf{n})$$

$$=(b_2c_3-b_3c_2)\,\mathbf{m}\times\mathbf{n}+(b_3c_1-b_1c_3)\,\mathbf{n}\times\mathbf{l}$$
$$+(b_1c_2-b_2c_1)\,\mathbf{l}\times\mathbf{m}.$$

$$\therefore \quad [\mathbf{abc}]=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

$$=[a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n}]\cdot[(b_2c_3-b_3c_2)\,\mathbf{m}\times\mathbf{n}$$
$$+(b_3c_1-b_1c_3)\,\mathbf{n}\times\mathbf{l}+(b_1c_2-b_2c_1)\,\mathbf{l}\times\mathbf{m}]$$

$$=a_1(b_2c_3-b_3c_2)\,\mathbf{l}\cdot(\mathbf{m}\times\mathbf{n})+a_2\,(b_3c_1-b_1c_3)\,\mathbf{m}\cdot(\mathbf{n}\times\mathbf{l})$$
$$+a_3(b_1c_2-b_2c_1)\mathbf{n}\cdot(\mathbf{l}\times\mathbf{m}).$$

Also other terms in the above product vanish as the scalar triple is zero when two of the vectors are equal.

Again we know that

$$\mathbf{l}\cdot(\mathbf{m}\times\mathbf{n})=\mathbf{m}\cdot(\mathbf{n}\times\mathbf{l})=\mathbf{n}\cdot(\mathbf{l}\times\mathbf{m})=[\mathbf{lmn}].$$

$$[\mathbf{abc}]=[a_1\,(b_2c_3-b_3c_2)+a_2\,(b_3c_1-b_1c_3)+a_3\,(b_1c_2-b_2c_1]\,[\mathbf{lmn}].$$

$$\therefore \quad [\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\mathbf{lmn}]. \qquad \textbf{(Agra 38)}$$

In case **a, b, c** be expressed in terms of unit vectors **i, j, k,** then $[\mathbf{ijk}]=1$ and hence

$$[\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \qquad \textbf{(Agra 40)}$$

Above is the well known expression for the volume of a parallelopiped whose one vertex is at the origin and the other three at (a_1, a_2, a_3) (b_1, b_2, b_3) and (c_1, c_2, c_3) [rectangular coordinates].

Note :—In case any two vectors are equal or parallel then the two rows in the above determinant will be identical and as such it will be zero and hence if two of the vectors **a**, **b**, **c** be either equal or parallel, then $[\mathbf{abc}]=0$.

Cor. 5. *To deduce the distributive law of cross product of two vectors by the help of scalar triple product, i. e.*

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

(Agra 51, Raj. 57)

Let **r** be any vector; then since scalar product is distributive, we have

$$\mathbf{r}\cdot[\mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}]$$

$$=\mathbf{r}\cdot[\mathbf{a}\times(\mathbf{b}+\mathbf{c})]-\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})-\mathbf{r}\cdot(\mathbf{a}\times\mathbf{c})\ldots \qquad \ldots(1)$$

Again we know that in scalar triple product the position of dot and cross can be changed without altering the value of the product. Hence we can write R. H. S. of (1), as $(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{b}+\mathbf{c})-(\mathbf{r}\times\mathbf{a})\cdot\mathbf{b}-(\mathbf{r}\times\mathbf{a})\cdot\mathbf{c}$
which can be written as

$$(\mathbf{r}\times\mathbf{a})\cdot[\mathbf{b}+\mathbf{c}-\mathbf{b}-\mathbf{c}]=(\mathbf{r}\times\mathbf{a})\cdot 0=0$$

because scalar product is distributive.

Hence we have for all values of **r**,

$$\mathbf{r}\cdot[\mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}]=0.$$

$$\therefore\ \mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}=0$$

or $$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

The other possibility of either **r** being zero or it being

perpendicular to $\mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}$ is ruled out by the fact that $\mathbf{r}$ is any vector whatsoever.

§ 6. Vector Triple Product.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\,\mathbf{c}.$$

(Alld. M. Sc. 1950, Pb. 60, Agra 32, 35, 59)

Note :—$\mathbf{a}\cdot\mathbf{c}$ is a scalar and occurs as a coefficient of vector $\mathbf{b}$ and similarly, $\mathbf{a}\cdot\mathbf{b}$ being scalar occurs as a coefficient of $\mathbf{c}$.

1st Method. Consider the vector triple product when two of the vectors are equal, *i. e.*

$$\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot\mathbf{b}\,\mathbf{a}-\mathbf{a}\cdot\mathbf{a}\,\mathbf{b}.$$

We have done that if there be a vector $\mathbf{b}$, then its component along a given vector $\mathbf{a}$ is $\dfrac{\mathbf{b}\cdot\mathbf{a}}{a^2}\,\mathbf{a}$ (Page 103)

or $$\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}\,;\quad\because\quad a^2=\mathbf{a}\cdot\mathbf{a}$$

and in a direction perpendicular to $\mathbf{a}$ its component is

$$-\frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{a^2}\quad\text{or}\quad-\frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{\mathbf{a}\cdot\mathbf{a}}\,;$$

(Page 120)

$$\therefore\quad \mathbf{b}=\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}-\frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{\mathbf{a}\cdot\mathbf{a}}.$$

Multiplying by scalar $\mathbf{a}\cdot\mathbf{a}$ and transposing, we get

$$\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot\mathbf{a}\,\mathbf{a}-\mathbf{a}\cdot\mathbf{a}\,\mathbf{b}=\mathbf{a}\cdot\mathbf{b}\,\mathbf{a}-\mathbf{a}\cdot\mathbf{a}\,\mathbf{b},$$

$$\because\quad \mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{b}.$$

Rule :—*First we take the dot product of the vector outside the bracket with the extreme vector inside the bracket and it becomes the coefficient of the remaining one. Then we take the dot product of the vector outside the bracket with the nearer one inside the bracket and it becomes the coefficient of the remaining one. The same rule is true when all the vectors are unequal.*

Again $(\mathbf{a}\times\mathbf{b})\times\mathbf{a}=-\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=-(\mathbf{a}\cdot\mathbf{b}\,\mathbf{a}-\mathbf{a}\cdot\mathbf{a}\,\mathbf{b})$

or $$(\mathbf{a}\times\mathbf{b})\times\mathbf{a}=\mathbf{a}\cdot\mathbf{a}\,\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\,\mathbf{a}$$

which obey the rule written above.

All the vectors being unequal.

Let $\mathbf{P}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times\mathbf{n}$ where $\mathbf{n}=\mathbf{b}\times\mathbf{c}$ which represents a vector perpendicular to the plane containing **b** and **c.**

$\therefore$ $\mathbf{P}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times\mathbf{n}$ is perpendicular to both **a** and **n** but **n** being perpendicular to the plane containing **b** and **c** and therefore **P** is perpendicular to **a** and it lies in the plane of **b** and **c**. Hence **P** is expressible in terms of **b** and **c.**

Let $$\mathbf{P}=x\mathbf{b}+y\mathbf{c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

$$\mathbf{P}\cdot\mathbf{a}=x\mathbf{a}\cdot\mathbf{b}+y\mathbf{a}\cdot\mathbf{c} \quad \ldots \quad \ldots \quad \ldots(2)$$

But we have already stated that **P** is perpendicular to **a.**

$\therefore$ $\mathbf{P}\cdot\mathbf{a}=0$; $\therefore$ from (2), $\dfrac{x}{\mathbf{a}\cdot\mathbf{c}}=\dfrac{-y}{\mathbf{a}\cdot\mathbf{b}}=k$ say.

Putting the values of x and y in (1), we get

$$\mathbf{P}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=k\,(\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\,\mathbf{c}) \quad \ldots \quad \ldots(3)$$

Now we have to find the value of k.

$$\therefore \quad \mathbf{P}\cdot\mathbf{b}=\mathbf{k}\,(\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\,\mathbf{b}\cdot\mathbf{c}) \ldots \quad \ldots(4)$$

Now $\mathbf{P}\cdot\mathbf{b}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\cdot\mathbf{b}$

$\qquad =\mathbf{a}\cdot[(\mathbf{b}\times\mathbf{c})\times\mathbf{b}]$

as the position of dot and cross can be changed at pleasure if the cyclic order is maintained

$=\mathbf{a}\cdot[\mathbf{b}\cdot\mathbf{b}\,\mathbf{c}-\mathbf{b}\cdot\mathbf{c}\,\mathbf{b}]$ by the rule written before when two vectors are equal

$=\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\,\mathbf{b}\cdot\mathbf{c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots(5)$

Equating the values of $\mathbf{P}\cdot\mathbf{b}$ from (4) and (5), we get $k=1$.

Hence from (3) by putting $k=1$, we get

$$\mathbf{P}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\,\mathbf{c}$$

or $$(\mathbf{b}\times\mathbf{c})\times\mathbf{a}=-\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$$

$$=\mathbf{a}\cdot\mathbf{b}\,\mathbf{c}-\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}.$$

The above obeys the same law as written before in the rule *i. e.* first we write the dot product of extremes and then of nearer ones ; the factor outside the bracket is included in both the dot products.

Note :—Just as $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ etc. but $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is not equal to $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$ as the former is coplanar with **b** and **c** and expressible in terms of **b** and **c** whereas the later is coplanar with **c** and **a** and is expressible in terms of **c** and **a**,

i. e. $$\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=\mathbf{b}\cdot\mathbf{a}\ \mathbf{c}-\mathbf{b}\cdot\mathbf{c}\ \mathbf{a}.$$

2nd Method.

Let $$\mathbf{P}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c}).$$

Consider unit vectors, **i**, **j**, **k** and let **b** be along **j**.

$\therefore$ $\mathbf{b}=b_2\mathbf{j}$ say and let **k** be perpendicular to **b** and in the plane of **b** and **c**.

$\therefore$ $\mathbf{c}=c_2\mathbf{j}+c_3\mathbf{k}$ and let the third vector **a** in **terms of i, j** and **k** be $a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$.

$$\therefore\quad \mathbf{b}\times\mathbf{c}=b_2\mathbf{j}\times(c_2\mathbf{j}+c_3\mathbf{k})=b_2c_3\mathbf{j}\times\mathbf{k}=b_2c_3\mathbf{i}.$$

$$\because\quad \mathbf{j}\times\mathbf{j}=0 \text{ and } \mathbf{j}\times\mathbf{k}=\mathbf{i},$$

$$\therefore\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_2c_3\mathbf{i})$$
$$=a_2b_2c_3\mathbf{j}\times\mathbf{i}+a_3b_2c_3\mathbf{k}\times\mathbf{i}$$
$$=a_3b_2c_3\mathbf{j}-a_2b_2c_3\mathbf{k}\quad\ldots\quad\ldots\quad\ldots(1)$$

$$\because\quad \mathbf{j}\times\mathbf{i}=-\mathbf{i}\times\mathbf{j}=-\mathbf{k} \text{ and } \mathbf{k}\times\mathbf{i}=\mathbf{j}.$$

Again $(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}=[(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(c_2\mathbf{j}+c_3\mathbf{k})]\,b_2\mathbf{j}$

$$=(a_2c_2+a_3c_3)\,b_2\mathbf{j}\quad\because\ \mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0.$$

$$(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=[(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot b_2\mathbf{j}]\,(c_2\mathbf{j}+c_3\mathbf{k})$$
$$=a_2b_2\,(c_2\mathbf{j}+c_3\mathbf{k}).$$

$$(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=a_3c_3b_2\mathbf{j}-a_2b_2c_3\mathbf{k}\quad\ldots\quad\ldots(2)$$

Hence from (1) and (2), we get

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c},$$

Similarly,

$$\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}.$$

$$\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}.$$ Hence proved.

Adding, we get $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$ because $(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}$ etc.

(Agra 42, 53, Annamalai 38, Andhra 36)

Note :—The component of $\mathbf{r}$ perpendicular to $\mathbf{a}$ was shown to be $\mathbf{r}-\dfrac{\mathbf{r}\cdot\mathbf{a}}{a^2}\cdot\mathbf{a}$ (See P. 103)

$$=\frac{a^2\mathbf{r}-\mathbf{r}\cdot\mathbf{a}\,\mathbf{a}}{a^2}=\frac{\mathbf{a}\cdot\mathbf{a}\,\mathbf{r}-\mathbf{r}\cdot\mathbf{a}\,\mathbf{a}}{a^2}$$ (See § 8 P. 120)

$$=\frac{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})}{\mathbf{a}^2}\quad\text{or}\quad=-\frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{r})}{\mathbf{a}^2}.$$

Note :—Verify the above formula

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$$

by taking $\mathbf{a}=\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{c}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

§ 7. Scalar product of four vectors **(Agra 37, 51)**

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$$

Let us suppose that $\mathbf{c}\times\mathbf{d}=\mathbf{n}$.

$$\therefore\quad(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{n}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{n})$$

as the position of dot and cross can be changed,

$$=\mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]=\mathbf{a}\cdot[\mathbf{b}\cdot\mathbf{d}\,\mathbf{c}-\mathbf{b}\cdot\mathbf{c}\,\mathbf{d}]$$

$$=\mathbf{b}\cdot\mathbf{d}\,\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}\,\mathbf{a}\cdot\mathbf{d}=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$$

§ 8. Vector product of four vectors $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$

Let $\mathbf{c}\times\mathbf{d}=\mathbf{n}$. **(Agra 38, 42)**

$$\therefore\quad(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\times\mathbf{b})\times\mathbf{n}=\mathbf{a}\cdot\mathbf{n}\,\mathbf{b}-\mathbf{b}\cdot\mathbf{n}\,\mathbf{a}$$

$$=\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})\,\mathbf{b}-\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})\,\mathbf{a}\ldots\quad\ldots\quad\ldots(1)$$

$$=[\mathbf{acd}]\,\mathbf{b}-[\mathbf{bcd}]\,\mathbf{a}.$$

Again let us put $(\mathbf{a}\times\mathbf{b})=\mathbf{m}$.

$$\therefore\quad(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{m}\times(\mathbf{c}\times\mathbf{d})=\mathbf{m}\cdot\mathbf{d}\,\mathbf{c}-\mathbf{m}\cdot\mathbf{c}\,\mathbf{d}$$

$$=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}\,\mathbf{c}-(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\,\mathbf{d}$$

$$=[\mathbf{abd}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{d}\quad\ldots\quad\ldots\quad\ldots(2)$$

Equating (1) and (2), we get

$$[\mathbf{acd}]\,\mathbf{b}-[\mathbf{bcd}]\,\mathbf{a}=[\mathbf{abd}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{d}$$

or $$[\mathbf{bcd}]\,\mathbf{a}-[\mathbf{acd}]\,\mathbf{b}-[\mathbf{abd}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{d}=0.$$

Now replacing **d** by **r**, we get

$$[\mathbf{abc}]\,\mathbf{r}=[\mathbf{bcr}]\,\mathbf{a}-[\mathbf{acr}]\,\mathbf{b}+[\mathbf{abr}]\,\mathbf{c} \quad \ldots \quad \ldots (3)$$

Now a scalar triple product remains unchanged if the cyclic order of the factors is maintained but for every change of cyclic order, there is a change of minus sign.

$$\therefore \quad [\mathbf{bcr}]=[\mathbf{rbc}],\ [\mathbf{acr}]=[\mathbf{rac}]=-[\mathbf{rca}]$$

and $$[\mathbf{abr}]=[\mathbf{rab}].$$

Hence from (3), we have

$$\mathbf{r}=\frac{[\mathbf{rbc}]\,\mathbf{a}+[\mathbf{rca}]\,\mathbf{b}+[\mathbf{rab}]\,\mathbf{c}}{[\mathbf{abc}]} \quad \ldots \quad \ldots (4)$$

(Agra 35, 46, 57, 60)

Above relation expressess a vector **r** in terms of any other three vectors **a, b, c** provided they are not coplanar *i.e.*, $[\mathbf{abc}]\neq 0$.

Rule. *The last three vectors in the numerator are the cyclic arrangement of* **a, b** *and* **c,** *i.e.,* **bca, cab** *and* **abc** *and the first being* **r.**

§ 9. Reciprocal system of vectors. **(Agra 59, 60)**

The three vectors **a′, b′, c′** defined by the equations

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$$

are called reciprocal system to the vectors **a, b, c** which are non-coplanar *i.e.* $[\mathbf{abc}]\neq 0$.

Property 1. If **a, b, c** and **a′, b′, c′** be reciprocal system of vectors, then $\mathbf{a}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{b}'=\mathbf{c}\cdot\mathbf{c}'=1$.

Now $$\mathbf{a}\cdot\mathbf{a}'=\mathbf{a}\cdot\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\frac{[\mathbf{abc}]}{[\mathbf{abc}]}=1.$$

Similarly, $$\mathbf{b}\cdot\mathbf{b}'=\mathbf{c}\cdot\mathbf{c}'=1.$$

$$\therefore \quad \mathbf{a}\cdot\mathbf{a}'+\mathbf{b}\cdot\mathbf{b}'+\mathbf{c}\cdot\mathbf{c}'=3$$

or $$\mathbf{a}'=\frac{1}{\mathbf{a}},\ \mathbf{b}'=\frac{1}{\mathbf{b}}\ \text{and}\ \mathbf{c}'=\frac{1}{\mathbf{c}}.$$

It is because of the above property that the two systems of vectors are called reciprocal systems.

Property 2. *The product of any vector of one system with a vector of the other system which does not correspond to it is zero, i.e.,* $\mathbf{a}\cdot\mathbf{b}'=0$.

$$\mathbf{a}\cdot\mathbf{b}'=\mathbf{a}\cdot\frac{[\mathbf{c}\times\mathbf{a}]}{[\mathbf{abc}]}=\frac{[\mathbf{aca}]}{[\mathbf{abc}]}=0$$

as the numerator is the scalar triple product of three vectors two of which are equal and hence it is zero. **(Cor. 2 P. 137)**

Similarly $\mathbf{a}\cdot\mathbf{c}'=\mathbf{b}\cdot\mathbf{c}'=\mathbf{c}\cdot\mathbf{a}'=0$ etc. etc.

Cor. *Thus we conclude from the two properties that if* $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ *be reciprocal system to* $\mathbf{a}, \mathbf{b}, \mathbf{c}$, *then* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *is a reciprocal system to* $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.

Property 3. *The scalar triple product* $[\mathbf{abc}]$ *of any three non-coplanar vectors is reciprocal to the corresponding scalar triple product formed out of the reciprocal system of vectors* $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.

(Agra 47, 51, 59)

i.e.

$$[\mathbf{a}'\mathbf{b}'\mathbf{c}']=\frac{1}{[\mathbf{abc}]}.$$

$$[\mathbf{a}'\mathbf{b}'\mathbf{c}']=\mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}') \qquad \ldots \quad \ldots(1)$$

Now substitute the values of $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ in terms of $\mathbf{a}$, $\mathbf{b}$ anc $\mathbf{c}$.

$$\therefore \quad [\mathbf{a}'\mathbf{b}'\mathbf{c}']=\frac{(\mathbf{b}\times\mathbf{c})\cdot[(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})]}{[\mathbf{abc}]^3} \text{ from (1)} \ldots \quad \ldots(2)$$

Now $(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\times\mathbf{a})\times\mathbf{m}$ say $=\mathbf{m}\cdot\mathbf{c}\,\mathbf{a}-\mathbf{m}\cdot\mathbf{a}\,\mathbf{c}$

$$=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\,\mathbf{a}-(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}\,\mathbf{c}=[\mathbf{abc}]\,\mathbf{a}.$$

$$\because \quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=0.$$

$$\therefore \quad [\mathbf{a}'\mathbf{b}'\mathbf{c}']=\frac{(\mathbf{b}\times\mathbf{c})\cdot[\mathbf{abc}]\,\mathbf{a}}{[\mathbf{abc}]^3}=\frac{[\mathbf{abc}]\,(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}}{[\mathbf{abc}]^3} \text{ from (2)}$$

$$=\frac{[\mathbf{abc}]\,[\mathbf{bca}]}{[\mathbf{abc}]^3}=\frac{[\mathbf{abc}]\,[\mathbf{abc}]}{[\mathbf{abc}]^3}=\frac{1}{[\mathbf{abc}]}.$$

$$\therefore \quad [\mathbf{abc}]\,[\mathbf{a}'\mathbf{b}'\mathbf{c}']=1. \qquad \textbf{(Agra 47)}$$

Cor. *From above, we conclude that*

$$[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2.$$

(Agra 36, 41, 51, 53, 57, 60; Pb. 60; Andhra 38; Benaras 56; Rajputana 56)

If **a**, **b**, **c** be expressed in terms of unit vectors, *i. e.* $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ etc., then we have already done that

$$[\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ **(Cor. 4 P. 138)**

Again $\mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$

$$=(a_2b_3-a_3b_2)\,\mathbf{i}+(b_1a_3-b_3a_1)\,\mathbf{j}+(a_1b_2-a_2b_1)\,\mathbf{k}.$$

(P. 121)

and similarly we can write for $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$.

$\therefore\ [\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]$

$$=\begin{vmatrix} a_2b_3-a_3b_2 & b_1a_3-b_3a_1 & a_1b_2-a_2b_1 \\ b_2c_3-b_3c_2 & c_1b_3-c_3b_1 & b_1c_2-b_2c_1 \\ c_2a_3-c_3a_2 & a_1c_3-a_3c_1 & c_1a_2-c_2a_1 \end{vmatrix}$$

(Cor. 4, P. 138)

$$=\begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}=\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

(where capital letters denote the co-factors of the corresponding small letters)

which is equal to

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2,\ i.\ e.\ [\mathbf{abc}]^2.$$

(Refer Author's Algebra)

Note. *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are non-coplanar i. e.* $[\mathbf{abc}] \neq 0$, *then* $[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}]$ *are also non-coplanar as their scalar triple product is* $[\mathbf{abc}]^2$ *which is not zero.*

Cor. We have done before that any vector $\mathbf{r}$ can be expressed in terms of three non-coplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

as $$\mathbf{r} = \frac{[\mathbf{rbc}]\,\mathbf{a} + [\mathbf{rca}]\,\mathbf{b} + [\mathbf{rab}]\,\mathbf{c}}{[\mathbf{abc}]} \qquad [\text{P. 144 (iv)}]$$

$$\frac{[\mathbf{rbc}]\,\mathbf{a}}{[\mathbf{abc}]} = \frac{\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]}\,\mathbf{a} = \mathbf{r} \cdot \mathbf{a}'\mathbf{a}$$

where $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ form a **reciprocal** system of vectors to $\mathbf{a}, \mathbf{b}, \mathbf{c}$; $\therefore \quad \mathbf{r} = \mathbf{r} \cdot \mathbf{a}'\mathbf{a} + \mathbf{r} \cdot \mathbf{b}'\mathbf{b} + \mathbf{r} \cdot \mathbf{c}'\mathbf{c}.$ **(Agra 38)**

Also the two systems of vectors $[\mathbf{abc}]$, $[\mathbf{a}'\mathbf{b}'\mathbf{c}']$, each is reciprocal of the other and as such any vector $\mathbf{r}$ can also be written as

$$\mathbf{r} = \mathbf{r} \cdot \mathbf{a}\,\mathbf{a}' + \mathbf{r} \cdot \mathbf{b}\,\mathbf{b}' + \mathbf{r} \cdot \mathbf{c}\,\mathbf{c}'.$$

Again $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $[\mathbf{i}\,\mathbf{j}\,\mathbf{k}] = 1$.

$\therefore$ the system of vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is its own reciprocal.

Hence in terms of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we have

$$\mathbf{r} = \mathbf{r} \cdot \mathbf{i}\,\mathbf{i} + \mathbf{r} \cdot \mathbf{j}\,\mathbf{j} + \mathbf{r} \cdot \mathbf{k}\,\mathbf{k}. \qquad [\text{See P. 104}]$$

Exercise

Ex. 1. *Prove that* $[\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}] = 2\,[\mathbf{abc}]$, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *being three vectors.* **(Dacca 27, 29)**

Ex. 2. *Prove that*

$$[\mathbf{lmn}]\,[\mathbf{abc}] = \begin{vmatrix} \mathbf{l} \cdot \mathbf{a} & \mathbf{l} \cdot \mathbf{b} & \mathbf{l} \cdot \mathbf{c} \\ \mathbf{m} \cdot \mathbf{a} & \mathbf{m} \cdot \mathbf{b} & \mathbf{m} \cdot \mathbf{c} \\ \mathbf{n} \cdot \mathbf{a} & \mathbf{n} \cdot \mathbf{b} & \mathbf{n} \cdot \mathbf{c} \end{vmatrix},$$

and give its cartesian equivalent.

(Agra 38, 40, 47, 49, 51, 54; Lucknow 52, 5 ; Pb. 60; Benaras 52; Annamalai 47)

Let $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ be a system of vectors reciprocal to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and hence $[\mathbf{a}'\mathbf{b}'\mathbf{c}']\,[\mathbf{abc}] = 1$.

Also we know that any vector $\mathbf{r}$ can be expressed in terms of any three non-coplanar vectors as

$$\mathbf{r}=\mathbf{r}\cdot\mathbf{a}\,\mathbf{a}'+\mathbf{r}\cdot\mathbf{b}\,\mathbf{b}'+\mathbf{r}\cdot\mathbf{c}\,\mathbf{c}'.$$

We shall express the vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ in terms of $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$.

$$\mathbf{l}=\mathbf{l}\cdot\mathbf{a}\,\mathbf{a}'+\mathbf{l}\cdot\mathbf{b}\,\mathbf{b}'+\mathbf{l}\cdot\mathbf{c}\,\mathbf{c}'$$

$$\mathbf{m}=\mathbf{m}\cdot\mathbf{a}\,\mathbf{a}'+\mathbf{m}\cdot\mathbf{b}\,\mathbf{b}'+\mathbf{m}\cdot\mathbf{c}\,\mathbf{c}'.$$

$$\mathbf{n}=\mathbf{n}\cdot\mathbf{a}\,\mathbf{a}'+\mathbf{n}\cdot\mathbf{b}\,\mathbf{b}'+\mathbf{n}\cdot\mathbf{c}\,\mathbf{c}'.$$

$$\therefore\quad [\mathbf{lmn}]=\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\cdot\mathbf{c}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\cdot\mathbf{c}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\cdot\mathbf{c}\end{vmatrix}[\mathbf{a}'\mathbf{b}'\mathbf{c}']$$

[Cor. 4 P. 138]

Multiply both sides by $[\mathbf{abc}]$ and since $[\mathbf{abc}]\,[\mathbf{a}'\mathbf{b}'\mathbf{c}']=1$, we get the required result.

Cartesian Equivalent.

Let $\mathbf{l}=l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k}$ etc.

$\mathbf{a}=a_1\mathbf{i}+\mathbf{a}_2\mathbf{j}+\mathbf{a}_3\mathbf{k}$ etc.

$$\therefore\quad [\mathbf{lmn}][\mathbf{abc}]=\begin{vmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_2\\ n_1 & n_2 & n_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\quad\ldots(1)$$

Also $\mathbf{l}\cdot\mathbf{a}=a_1l_1+a_2l_2+a_3l_3$, $\because\quad \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1$.

$$\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\cdot\mathbf{c}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\cdot\mathbf{c}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\cdot\mathbf{c}\end{vmatrix}=\begin{vmatrix}a_1l_1+a_2l_2+a_3l_3 & \Sigma b_1l_1 & \Sigma c_1l_1\\ a_1m_1+a_2m_2+a_3m_3 & \Sigma b_1m_1 & \Sigma c_1m_1\\ a_1n_1+a_2n_2+a_3n_3 & \Sigma b_1n_1 & \Sigma c_1n_1\end{vmatrix}\quad\ldots(2)$$

$$=\begin{vmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Note :—In case **l**, **m**, **n** are same as **a**, **b**, **c**, then we get

$$[\mathbf{abc}]\,[\mathbf{abc}]=\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}=[\mathbf{abc}]^2$$

or $=[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]$ as proved in **Cor.** P. 146.

Note. 2. In case **l**, **m**, **n** are reciprocal to **a**, **b**, **c** then $[\mathbf{lmn}]\,[\mathbf{abc}]=1$ and $\mathbf{l}\cdot\mathbf{a}=\mathbf{m}\cdot\mathbf{b}=\mathbf{n}\cdot\mathbf{c}=1$ but $\mathbf{l}\cdot\mathbf{b}=\mathbf{l}\cdot\mathbf{c}=0$, $\mathbf{m}\cdot\mathbf{a}=\mathbf{m}\cdot\mathbf{c}=0$, $\mathbf{n}\cdot\mathbf{a}=\mathbf{n}\cdot\mathbf{b}=0$. [Prop. 2. P. 145].

(Benares 55)

$\therefore$ we get $1=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$

Ex. 3. *Prove independently that*

$$[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2=\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

The first part is proved on page 145.

Now we know that $(\mathbf{a}\times\mathbf{b})^2=a^2b^2\sin^2\theta=a^2b^2(1-\cos^2\theta)$

$$=a^2b^2-a^2b^2\cos^2\theta$$
$$=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\cdot\mathbf{b})^2.$$

$\because$ square of a vector is square of its module and it stands for the dot product of a vector by itself *i.e.* $a^2=\mathbf{a}^2=\mathbf{a}\cdot\mathbf{a}$ and also $\mathbf{a}\cdot\mathbf{b}=ab\cos\theta$.

Now $[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]^2=[\mathbf{m}\times\mathbf{c}]^2=\mathbf{m}^2\mathbf{c}^2-(\mathbf{m}\cdot\mathbf{c})^2$

$$=(\mathbf{a}\times\mathbf{b})^2\,\mathbf{c}^2-[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]^2$$
$$=[\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\cdot\mathbf{b})^2]\,\mathbf{c}^2-[\mathbf{abc}]^2$$
$$=\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2-(\mathbf{a}\cdot\mathbf{b})^2\mathbf{c}^2-[\mathbf{abc}]^2\ \ldots\quad\ldots(1)$$

Again $[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]=(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}$,

$\therefore$ $[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]^2=[(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}]^2$

$$=(\mathbf{c}\cdot\mathbf{a})^2\mathbf{b}^2+(\mathbf{c}\cdot\mathbf{b})^2\mathbf{a}^2-2\,(\mathbf{c}\cdot\mathbf{a})\,(\mathbf{c}\cdot\mathbf{b})\,(\mathbf{b}\cdot\mathbf{a})\ldots\quad\ldots(2)$$

Equating the values of $[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]^2$ from (1) and (2), we get $\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2-(\mathbf{a}\cdot\mathbf{b})^2\,\mathbf{c}^2-[\mathbf{abc}]^2$

$$=(\mathbf{c}\cdot\mathbf{a})^2\,\mathbf{b}^2+(\mathbf{c}\cdot\mathbf{b})^2\,\mathbf{a}^2-2\,(\mathbf{c}\cdot\mathbf{a})\,(\mathbf{c}\cdot\mathbf{b})\,(\mathbf{b}\cdot\mathbf{a}).$$

$$\therefore\quad [\mathbf{abc}]^2=\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2-(\mathbf{a}\cdot\mathbf{b})^2\,\mathbf{c}^2-(\mathbf{b}\cdot\mathbf{c})^2\,\mathbf{a}^2-(\mathbf{c}\cdot\mathbf{a})^2\,\mathbf{b}^2+2\,(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{c}\cdot\mathbf{a}).$$

Now if we expand the determinant on L. H. S., we get the same result. Hence proved.

Ex. 4. *Prove that if* **l**, **m**, **n** *be three non-coplanar vectors*

$$[\mathbf{lmn}]\,(\mathbf{a}\times\mathbf{b})=\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\end{vmatrix}.$$

(Agra 49, Dacca 40)

Expressing **l**, **m**, **n** and **a**, **b** in terms of unit vectors **i**, **j**, **k**, $\mathbf{l}=l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k}$

and $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}.$

$$\therefore\quad \mathbf{l}\cdot\mathbf{a}=l_1a_1+l_2a_2+l_3a_3 \text{ etc.}$$

and $[\mathbf{lmn}]=\begin{vmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}$ and $(\mathbf{a}\times\mathbf{b})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$

$\therefore\quad [\mathbf{lmn}]\,(\mathbf{a}\times\mathbf{b})$

$$=\begin{vmatrix}l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k} & l_1a_1+l_2a_2+l_3a_3 & l_1b_1+l_2b_2+l_3b_3\\ m_1\mathbf{i}+m_2\mathbf{j}+m_3\mathbf{k} & m_1a_1+m_2a_2+m_3a_3 & m_1b_1+m_2b_2+m_3b_3\\ n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k} & n_1a_1+n_2a_2+n_3a_3 & n_1b_1+n_2b_2+n_3b_3\end{vmatrix}$$

$$=\begin{vmatrix}\mathbf{l} & \mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b}\\ \mathbf{m} & \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b}\\ \mathbf{n} & \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b}\end{vmatrix}=\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\end{vmatrix}.$$

Ex. 5. *If* **a**, **b**, **c** *are three non-coplanar vectors, then prove the following* :—

1. $[\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b}]=[\mathbf{abc}]^2$.

(Agra 36, 41, 51, 53, 57, 60, Benaras 56, Andhra 38, Rajputana 56)

2. $\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b}$ *are also non-coplanar.*

3. Express **a**, **b**, **c** *in terms of* $\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b}$.

4. Express $\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b}$ *in terms of* **a**, **b**, **c**.

The first two parts we have already proved in Cor. P. 146 and Note P. 147.

3. Let $\mathbf{a}=l\,(\mathbf{b}\times\mathbf{c})+m\,(\mathbf{c}\times\mathbf{a})+n\,(\mathbf{a}\times\mathbf{b})$ ... ...(1)

Multiplying both sides scalarly by **a**,

$$\mathbf{a}\cdot\mathbf{a}=l\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+m\,\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})+n\,\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$$

or $$\mathbf{a}\cdot\mathbf{a}=l\,[\mathbf{abc}].$$

$\therefore$ scalar triple product is zero when two vectors are equal.

$$\therefore\quad l=\frac{\mathbf{a}\cdot\mathbf{a}}{[\mathbf{abc}]}.$$

Similarly multiplying both sides of (1) scalarly by **b** and **c**, we get

$$m=\frac{\mathbf{a}\cdot\mathbf{b}}{[\mathbf{abc}]}\text{ and }\mathbf{n}=\frac{\mathbf{a}\cdot\mathbf{c}}{[\mathbf{abc}]}.$$

$$\because\quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=[\mathbf{abc}].$$

Substituting the values of l, m and n in (1), we get

$$\mathbf{a}=\frac{\mathbf{a}\cdot\mathbf{a}}{[\mathbf{abc}]}(\mathbf{b}\times\mathbf{c})+\frac{\mathbf{a}\cdot\mathbf{b}}{[\mathbf{abc}]}(\mathbf{c}\times\mathbf{a})+\frac{\mathbf{a}\cdot\mathbf{c}}{[\mathbf{abc}]}(\mathbf{a}\times\mathbf{b}).$$

Similarly we can write the value of **b** and **c**.

4. Let $(\mathbf{b}\times\mathbf{c})=l\mathbf{a}+m\mathbf{b}+n\mathbf{c}$ ... ...(2)

Multiply both sides of (2) scalarly by $(\mathbf{b}\times\mathbf{c})$.

$$\therefore\quad (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c})=l\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+m\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})+n\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}).$$

$$\therefore\quad \text{as before } l=\frac{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{abc}]}.$$

Similarly multiply both sides of (2) scalarly by $(\mathbf{c}\times\mathbf{a})$ and $(\mathbf{a}\times\mathbf{b})$ and find $m=\dfrac{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{abc}]}$

and $$n=\frac{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{abc}]}.$$

Substituting the values of l, m, n in (2), we get the required result. Similarly we can express $\mathbf{c}\times\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Ex. 5. *Express a vector* $\mathbf{r}$ *as a linear combination of a vector* $\mathbf{a}$ *and another vector perpendicular to* $\mathbf{a}$ *and coplanar with* $\mathbf{r}$ *and* $\mathbf{a}$.

We have already stated that the vector $\mathbf{a}\times(\mathbf{a}\times\mathbf{r})$ is a vector perpendicular to $\mathbf{a}$ and coplanar with $\mathbf{a}$ and $\mathbf{r}$ and as such its dot product with $\mathbf{a}$ is xero, $\because$ dot product of two perpendicular vectors is zero.

Let $$\mathbf{r}=l\,\mathbf{a}+m\,\mathbf{a}\times(\mathbf{a}\times\mathbf{r}) \qquad \ldots \qquad \ldots(1)$$

Multiplying both sides scalarly by $\mathbf{a}$, we get

$$\mathbf{r}\cdot\mathbf{a}=l\,\mathbf{a}\cdot\mathbf{a}+m\,\mathbf{a}\cdot[\mathbf{a}\times(\mathbf{a}\times\mathbf{r})]=l\,\mathbf{a}\cdot\mathbf{a}$$

$$\therefore \quad l=\frac{\mathbf{r}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}.$$

Again multiply both sides of (1) vectorially by $\mathbf{a}$.

$$\therefore \quad \mathbf{r}\times\mathbf{a}=l\,\mathbf{a}\times\mathbf{a}+m\,[\mathbf{a}\times(\mathbf{a}\times\mathbf{r})]\times\mathbf{a}$$

$$=0+m\,[\mathbf{a}\cdot\mathbf{r}\,\mathbf{a}-\mathbf{a}\cdot\mathbf{a}\,\mathbf{r}]\times\mathbf{a}.$$

$$\mathbf{r}\times\mathbf{a}=m\,[\mathbf{a}\cdot\mathbf{r}\,(\mathbf{a}\times\mathbf{a})-\mathbf{a}\cdot\mathbf{a}\,(\mathbf{r}\times\mathbf{a})]=-m\,(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{r}\times\mathbf{a}).$$

$$\therefore \quad m=-\frac{1}{\mathbf{a}\cdot\mathbf{a}}.$$

Substituting the value of l and m in (1), we get

$$\mathbf{r}=\frac{\mathbf{r}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}-\frac{1}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}\times(\mathbf{a}\times\mathbf{r}).$$

Above shows that the component of a vector $\mathbf{r}$ along

a given direction $\mathbf{a}$ is $\dfrac{\mathbf{r}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$ and in a direction perpendicular to it is $-\dfrac{\mathbf{a}\times(\mathbf{a}\times\mathbf{r})}{\mathbf{a}\cdot\mathbf{a}}$.

[See § 8 P. 121]

Ex. 6. *Prove that*

$$\begin{aligned}
&[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}]\\
&\quad=[\mathbf{abd}]\,[\mathbf{cef}]-[\mathbf{abc}]\,[\mathbf{def}]\\
&\quad=[\mathbf{abe}]\,[\mathbf{fcd}]-[\mathbf{abf}]\,[\mathbf{ecd}]\\
&\quad=[\mathbf{cda}]\,[\mathbf{bef}]-[\mathbf{cdb}]\,[\mathbf{aef}].
\end{aligned}$$

(Agra 36, 61)

We know that any scalar triple product $[\mathbf{pqr}]$ is equal to $\mathbf{p}\cdot(\mathbf{q}\times\mathbf{r})=\mathbf{q}\cdot(\mathbf{r}\times\mathbf{p})=\mathbf{r}\cdot(\mathbf{p}\times\mathbf{q})$. Writing the given scalar triple product in the above three ways maintaining the cyclic order we shall get the three results as given. We shall show only one and the rest can be done by the students themselves.

$$\begin{aligned}
&[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}]\\
&\quad=(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{c}\times\mathbf{d})\times(\mathbf{e}\times\mathbf{f})]\\
&\quad=(\mathbf{a}\times\mathbf{b})\cdot[\mathbf{n}\times(\mathbf{e}\times\mathbf{f})]\\
&\quad=(\mathbf{a}\times\mathbf{b})\cdot[\mathbf{n}\cdot\mathbf{f}\ \mathbf{e}-\mathbf{n}\cdot\mathbf{e}\ \mathbf{f}]\\
&\quad=(\mathbf{n}\cdot\mathbf{f})\,[\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}]-(\mathbf{n}\cdot\mathbf{e})\,[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{f}]\\
&\quad=(\mathbf{c}\times\mathbf{d})\cdot\mathbf{f}\,[\mathbf{abe}]-(\mathbf{c}\times\mathbf{d})\cdot\mathbf{e}\,[\mathbf{abf}]\\
&\quad=[\mathbf{abe}]\,[\mathbf{fcd}]\cdot[\mathbf{2bf}]\,[\mathbf{ecd}].
\end{aligned}$$

Ex. 7. *Prove that* $[\mathbf{a}\times\mathbf{p},\ \mathbf{b}\times\mathbf{q},\ \mathbf{c}\times\mathbf{r}]+[\mathbf{a}\times\mathbf{q},\ \mathbf{b}\times\mathbf{r},\ \mathbf{c}\times\mathbf{p}]+[\mathbf{a}\times\mathbf{r},\ \mathbf{b}\times\mathbf{p},\ \mathbf{c}\times\mathbf{q}]=0$. **(Agra 34, 48, 59; Luck. 55)**

Expand first bracket as $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, second as $\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$ and third as $\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ and then add keeping in view that scalar triple is unchanged if cyclic order is maintained and its sign is changed for every change of cyclic order.

Ex. 8. *If* $\mathbf{a},\ \mathbf{b},\ \mathbf{c}$ *and* $\mathbf{a}',\ \mathbf{b}',\ \mathbf{c}'$ *form reciprocal system of vectors, then prove the following :—*

(1) $\mathbf{a}\cdot\mathbf{a}'+\mathbf{b}\cdot\mathbf{b}'+\mathbf{c}\cdot\mathbf{c}'=3.$

(2) $\mathbf{a}\times\mathbf{a}'+\mathbf{b}\times\mathbf{b}'+\mathbf{c}\times\mathbf{c}'=0.$

(3) $\mathbf{a}=\dfrac{\mathbf{b}'\times\mathbf{c}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']},\ \mathbf{b}=\dfrac{\mathbf{c}'\times\mathbf{a}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']},\ \mathbf{c}=\dfrac{\mathbf{a}'\times\mathbf{b}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']}.$

Ex. 9. *Find the value of*

$\mathbf{P}=\mathbf{i}\times(\mathbf{a}+\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k})$

$\mathbf{P}=(\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i}+(\mathbf{j}\cdot\mathbf{j})\,\mathbf{a}-(\mathbf{j}\cdot\mathbf{a})\,\mathbf{j}+(\mathbf{k}\cdot\mathbf{k})\,\mathbf{a}-(\mathbf{k}\cdot\mathbf{a})$

$=(\mathbf{a}+\mathbf{a}+\mathbf{a})-[(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{a}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{a}\cdot\mathbf{k})\,\mathbf{k}]$

$=3\mathbf{a}-\mathbf{a}=2\mathbf{a}.$ [§ 2·11 P. 106]

Ex. 10. *Prove the relation*

$\mathbf{a}\times[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]=\mathbf{b}\cdot\mathbf{d}\ \mathbf{a}\times\mathbf{c}-\mathbf{b}\cdot\mathbf{c}\ \mathbf{a}\times\mathbf{d}$

$=[\mathbf{acd}]\,\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\ \mathbf{c}\times\mathbf{d}$

and hence we prove that

$\mathbf{a}\times[\mathbf{b}\times\{\mathbf{c}\times(\mathbf{d}\times\mathbf{e})\}]$

$=[\mathbf{a}\cdot\mathbf{d}\ \mathbf{c}\cdot\mathbf{e}-\mathbf{c}\cdot\mathbf{d}\ \mathbf{a}\cdot\mathbf{e}]\,\mathbf{b}+(\mathbf{a}\cdot\mathbf{b})\,[\mathbf{c}\cdot\mathbf{d}\ \mathbf{e}-\mathbf{c}\cdot\mathbf{e}\ \mathbf{d}]$

(Agra 37, 42, 55; Delhi 51)

L.H.S. $=\mathbf{a}\times[\mathbf{b}\cdot\mathbf{d}\ \mathbf{c}-\mathbf{b}\cdot\mathbf{c}\ \mathbf{d}]$ etc.

or $=\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})\,\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\ \mathbf{c}\times\mathbf{d}.$

Ex. 11. *Prove that*

$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=0$

and dednce that

$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

$=\frac{1}{2}(\cos 2B-\cos 2A)=\cos^2 B-\cos^2 A$

and $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B.$

(Agra 37, 46, 50, 53, 60; Benaras 53; Delhi, Luck. 55, Allahabad M. Sc. 60)

$$=\begin{vmatrix}\mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{d}\\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{d}\end{vmatrix}+\begin{vmatrix}\mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{d}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{d}\end{vmatrix}+\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$$

[§ 7 P. 143

$=(\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d})-(\mathbf{c}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{d})+(\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}$

$+(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$

Now using the fact that $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ etc., we get

L. H. S.$=0$.

Let $\angle AOC = A$

and $\angle AOB = B$.

$\therefore$ $\angle BOC = A - B$.

Let $\angle BOD = A$.

$\therefore$ $\angle COD$

$= \angle BOD - \angle BOC$

$= A - (A - B) = B$.

$\therefore$ $\angle AOD$

$$= \angle AOC + \angle COD = A + B.$$

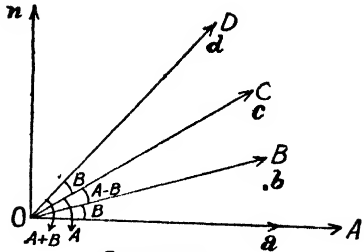


Fig No 81

Let $\mathbf{n}$ be a unit vector perpendicular to the plane of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ which are assumed to be coplanar.

Now we know that $(\mathbf{a} \times \mathbf{b}) = ab \sin \theta \, \mathbf{n}$ where θ is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$ measured from $\mathbf{a}$ in anti-chockwise direction towards $\mathbf{b}$.

Again we have proved that

$$(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 0$$

or $$(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) - (\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 0$$

or $$(bc \sin EOC)\, \mathbf{n} \cdot (ad \sin AOD)\, \mathbf{n} - (ac \sin AOC)\, \mathbf{n} \cdot (bd \sin BOD)\, \mathbf{n} + (ab \sin AOB)\, \mathbf{n} \cdot (cd \sin COD)\, \mathbf{n} = 0$$

or $$abcd \sin (A - B) \sin (A + B) - abcd \cdot \sin A \cdot \sin A + abcd \cdot \sin B \cdot \sin B = 0$$

$\because$ $\mathbf{n} \cdot \mathbf{n} = 1$.

$\therefore$ $\sin (A - B) \sin (A + B) = \sin^2 A - \sin^2 B$

$$= \frac{1 - \cos 2A}{2} - \frac{1 - \cos 2B}{2} = \cos 2B - \cos 2A.$$

Again $$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d}) \end{vmatrix}$$

$$= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$$

or $\quad (ab \sin B)\,\mathbf{n} \cdot (cd \sin B)\,\mathbf{n}=(ac \cos A)\,(bd \cos A)$
$$-\{bc \cos (A-B)\}\,\{ad \cos (A+B)\}.$$

Cancel $abcd$ and put $\mathbf{n} \cdot \mathbf{n}=1$ as $\mathbf{n}$ is a unit vector.

or $$\sin^2 B=\cos^2 A-\cos (A-B) \cos (A+B)$$

or $$\cos (A+B) \cos (A-B)=\cos^2 A-\sin^2 B$$
$$=\cos^2 B-\sin^2 A.$$

Ex. 12. *Prove that*

$$(\mathbf{a}\times\mathbf{b}) \cdot \;\mathbf{a}\times\mathbf{c})+(\mathbf{a}\times\mathbf{b}) \cdot (\mathbf{a}\times\mathbf{c})=(\mathbf{a} \cdot \mathbf{a})\,(\mathbf{b} \cdot \mathbf{c}).$$

(Annamalai 39)

Ex. 13. *Prove that*

$$2\,(\mathbf{c}\times\mathbf{d})\times(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} \mathbf{a} & a_1 & a_2 & a_3 \\ \mathbf{b} & b_1 & b_2 & b_3 \\ -\mathbf{c} & c_1 & c_2 & c_3 \\ -\mathbf{d} & d_1 & d_2 & d_3 \end{vmatrix}$$

where $\quad \mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k};\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ *etc.*

$$(\mathbf{c}\times\mathbf{d})\times(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\times\mathbf{d})\times\mathbf{m}=(\mathbf{m} \cdot \mathbf{c})\,\mathbf{d}-(\mathbf{m} \cdot \mathbf{d})\,\mathbf{c}$$
$$=[(\mathbf{a}\times\mathbf{b}) \cdot \mathbf{c}]\,\mathbf{d}-[(\mathbf{a}\times\mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}$$
$$=[\mathbf{abc}]\,\mathbf{d}-[\mathbf{abd}]\,\mathbf{c}$$
$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\mathbf{d}-\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\mathbf{c} \quad \ldots(1)$$

[Cor. 4 P. 138]

Aagain putting $(\mathbf{c}\times\mathbf{d})=\mathbf{n}$ and proceeding as above,

$$(\mathbf{c}\times\mathbf{d})\times(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\mathbf{a}-\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\mathbf{b} \quad \ldots(2)$$

Adding, we get

$$2(\mathbf{c}\times\mathbf{d})\times(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} \mathbf{a} & a_1 & a_2 & a_3 \\ \mathbf{b} & b_1 & b_2 & b_3 \\ -\mathbf{c} & c_1 & c_2 & c_3 \\ -\mathbf{d} & d_1 & d_2 & d_3 \end{vmatrix}.$$

$\because$ If we expand the above determinant, we get the four determinants of (1) and (2).

Ex. 14. *Prove that*

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})+(\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b})+(\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c})=2\,[\mathbf{bdc}]\,\mathbf{a}.$$

(Andhra 38)

Expand 1st in terms of **c** and **d**, 2nd terms of **a** and **c** and 3rd in terms of **a** and **d** etc.

Ex. 15. *Show that the perpendicular distance of a point C from the straight line through A and B is* $|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}|\div|\mathbf{b}-\mathbf{a}|$ *where* **a**, **b**, **c** *are the position vectors of A, B and C.*

The vector area of a triangle ABC is

$$\tfrac{1}{2}(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}). \qquad \text{[Ex. 6 P. 128]}$$

Its module is $\frac{1}{2}$ base $AB\times$ perpendicular from C on AB

$$=\tfrac{1}{2}\,|\mathbf{b}-\mathbf{a}|\times p.$$

$$\therefore\ p=|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|\div|\mathbf{b}-\mathbf{a}|.$$

CHAPTER IV

THE PLANE AND SPHERE

1. Vector equation of a plane. **(Agra 31, 39)**

Let there be a unit vector $\hat{\mathbf{n}}$ normal to the plane. If p be the length of the perpendicular from origin on the plane, then $\overrightarrow{ON}=p.\hat{\mathbf{n}}$.

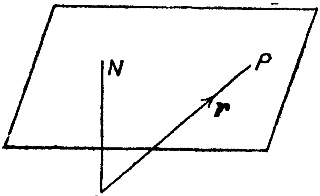


Fig. No. 82

Let $\mathbf{r}$ be the position vector of any point P on the plane ; then the projection of OP on ON is p.

But projection of OP on ON is $r\cos\theta=r\,.\,1\,.\cos\theta=\mathbf{r}\,.\,\hat{\mathbf{n}}$ and it being equal to p, we have the required equation of the plane as $\mathbf{r}\,.\,\hat{\mathbf{n}}=p$ ... ...(1)

p standing for the length of the perpendicular from the origin.

Cartesian form :—

Let the coordinates of P referred to unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ through O be x, y, z so that

$$\overrightarrow{OP}=\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$$

If l, m, n be the direction cosines of normal, then

$$\hat{\mathbf{n}}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}.$$

$$\therefore \quad \mathbf{r}\,.\,\hat{\mathbf{n}}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\,.\,(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=p$$

or $\qquad lx+my+nz=p\,, \quad \because \quad \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1$

and $\qquad \mathbf{i}\,.\,\mathbf{j}=\mathbf{j}\,.\,\mathbf{k}=\mathbf{k}\,.\,\mathbf{i}=0.$

which is the standard equation of the plane in Coordinate Geometry.

Cor. *In case there be any vector* **n** *parallel to unit vector* $\hat{\mathbf{n}}$ *and of module n, then* $\mathbf{n} = n \cdot \hat{\mathbf{n}}$.

Multiplying both sides of (1) by n, we get

$n(\mathbf{r} \cdot \hat{\mathbf{n}}) = np$ or $\mathbf{r} \cdot \mathbf{n} = np = q$, say, where $p = q/n$.

Thus $\mathbf{r} \cdot \mathbf{n} = q$ represents a plane; the length of the perpendicular from the origin is obtained by dividing the R. H. S. by the module of n,

i. e.
$$p = \frac{q}{n} = \frac{\text{R. H. S.}}{\text{Module of } \mathbf{n}}.$$

Cartesian to vector. If the plane be $2x+3y+4z=10$, then the corresponding vector equation is evidently $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) \cdot (2\mathbf{i}+3\mathbf{j}+4\mathbf{k}) = 10$ *i. e.* $\mathbf{r} \cdot \mathbf{n} = q$ and the length of perpendicular from origin is $\dfrac{q}{\text{module of } \mathbf{n}} = \dfrac{10}{\sqrt{(2^2+3^2+4^2)}}$

$$= \frac{10}{\sqrt{29}}.$$

Cor. 2. The equation of a plane that passes through a given point.

Let the position vector of a point A be **a** through which the plane passes and whose normal is **n**. If **r** be the position vector of any point in the plane, then AP lies in this plane and as such **n** is perpendicular to AP.

Therefore $\qquad \overrightarrow{AP} \cdot \mathbf{n} = 0$

or $\qquad (\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ or $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = q$ say.

The length of the perpendicular from origin on the

plane is

$$\frac{q}{|\mathbf{n}|}=\frac{\mathbf{a}\cdot\mathbf{n}}{|\mathbf{n}|}=\frac{\mathbf{a}\cdot\mathbf{n}}{n}=\mathbf{a}\cdot\hat{\mathbf{n}} \qquad \left[\because\ \frac{\mathbf{n}}{n}=\hat{\mathbf{n}}\right]$$

$=a\cdot 1\cos\theta$, *i.e.* the projection of OA along the normal.

Converse : To prove that the equation $\mathbf{r}\cdot\mathbf{n}=q$ represents a plane.

$$\mathbf{r}\cdot\mathbf{n}=q \qquad \ldots \quad \ldots(1)$$

Let **a** and **b** be the position vectors of any two points A and B which satisfy (1).

$$\therefore\quad \mathbf{a}\cdot\mathbf{n}=q \qquad \ldots \quad \ldots(2)$$

and

$$\mathbf{b}\cdot\mathbf{n}=q \qquad \ldots \quad \ldots(3)$$

Multiplying (2) by m and (3) by n and adding, we get

$$(m\mathbf{a}+n\mathbf{b})\cdot\mathbf{n}=(m+n)\,q$$

or

$$\frac{m\mathbf{a}+n\mathbf{b}}{m+n}\cdot\mathbf{n}=q \qquad \ldots \quad \ldots(4)$$

Equation (4) shows that the point whose position vector is $\dfrac{m\mathbf{a}+n\mathbf{b}}{m+n}$ also satisfies (1).

Now $\dfrac{m\mathbf{a}+n\mathbf{b}}{m+n}$ is any point on the line joining **a** and **b** and divides it in the ratio $n:m$. Thus we observe that every point on the line joining A and B satisfies equation (1) which therefore should represent a plane.

Cor. 3. Two sides of a plane : *The points whose position vectors are* **a** *and* **b** *lie on the same or opposite sides of a plane* $\mathbf{r}\cdot\mathbf{n}=q$ *according as* $\mathbf{a}\cdot\mathbf{n}-q$ *and* $\mathbf{b}\cdot\mathbf{n}-q$ *are of the same or opposite signs.*

Let the line joining A and B intersect the plane at P which divides AB in the ratio $n:m$ and therefore its

position vector is $\dfrac{m\mathbf{a}+n\mathbf{b}}{m+n}$. Since this point lies on the plane $\mathbf{r}\cdot\mathbf{n}=q$,

$$\therefore \quad \frac{m\mathbf{a}+n\mathbf{b}}{m+n}\cdot\mathbf{n}=q$$

or $$m(\mathbf{a}\cdot\mathbf{n}-q)=-n(\mathbf{b}\cdot\mathbf{n}-q)$$

or $$\frac{n}{m}=-\frac{\mathbf{a}\cdot\mathbf{n}-q}{\mathbf{b}\cdot\mathbf{n}-q}.$$

If n/m is +ive, *i.e.* $\mathbf{a}\cdot\mathbf{n}-q$ and $\mathbf{b}\cdot\mathbf{n}-q$ are of **opposite signs**, then P divides internally the join of A and B, *i e.* A and B are on the **opposite sides of the plane.**

If n/m is −ive, *i.e.* $\mathbf{a}\cdot\mathbf{n}-q$ and $\mathbf{b}\cdot\mathbf{n}-q$ are of **same sign** then P divides the join of A and B externally, *i.e.* A and B are on the **same side of the plane.** Hence Proved.

Ex. 1. *Find the equation of the plane through the point* $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ *and perpendicular to the vector* $3\mathbf{i}+4\mathbf{j}+7\mathbf{k}$. *Determine the perpendicular distance of this plane from origin.*

Here $\mathbf{n}=3\mathbf{i}+4\mathbf{j}+7\mathbf{k}$ and $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

$\therefore$ $\mathbf{a}\cdot\mathbf{n}=6+12-7=11$ and $n=\sqrt{(3^2+4^2+7^2)}=8\ldots(1)$

Now equation of the plane through a point $\mathbf{a}$ is

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0 \quad\text{or}\quad \mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}. \qquad \textbf{[Cor. 2]}$$

or $$\mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}+7\mathbf{k})=11. \qquad \text{[from (1)]}$$

Also $$p=\frac{\mathbf{a}\cdot\mathbf{n}}{|\mathbf{n}|}=\frac{11}{8}. \qquad \text{[from (1)]}$$

In cartesian form its equation is $3x+4y+7z=11$.

§ 2. Equation of plane satisfying the given conditions.

We have already seen that the equation of a plane is $\mathbf{r}\cdot\mathbf{n}=q$ and a plane through a given point is $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$, *i.e.* $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$.

Here we shall deduce the equation of plane satisfying other given conditions by the help of vector and scalar triple product.

I. Equation of a plane through three given points. **(Agra 51)**

Let the position vectors of any three points A, B and C through which the plane passes be **a**, **b** and **c** respectively. Let **r** be the position vector of any point P on the plane. Now the points P, A, B, C all lie on the same plane, *i.e.*, vectors $\overrightarrow{PA}$, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ *i.e.* $\mathbf{r}-\mathbf{a}$, $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{b}$ are coplanar. Now we know that if three vectors are coplanar, their scalar triple product vanishes. **[Cor. 1 P. 132]**

$$\therefore \quad (\mathbf{r}-\mathbf{a}) \cdot \{(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})\}=0$$

or $$(\mathbf{r}-\mathbf{a}) \cdot [\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}]=0, \quad \because \quad \mathbf{b}\times\mathbf{b}=0$$

or $$\mathbf{r} \cdot [\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}]$$
$$=\mathbf{a} \cdot (\mathbf{b}\times\mathbf{c})+\mathbf{a} \cdot (\mathbf{c}\times\mathbf{a})+\mathbf{a} \cdot (\mathbf{a}\times\mathbf{b}).$$
$$\therefore \quad \mathbf{r} \cdot [\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}]=[\mathbf{abc}] \quad \ldots \quad \ldots(1)$$

$\because \quad -\mathbf{a}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$ and $\mathbf{a} \cdot (\mathbf{c}\times\mathbf{a})=0=\mathbf{a} \cdot (\mathbf{a}\times\mathbf{b})$,

$\because$ scalar triple product is zero when two vectors are equal. **[Cor. 2 P. 132]**

The equation (1) is of the form $\mathbf{r} \cdot \mathbf{n}=q$ and is therefore the required equation of the plane.

The plane is clearly perpendicular to

$$\mathbf{n}=\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$$

which is equal to $2\overrightarrow{\triangle ABC}$ **(Q. 6 P. 128)**. If p be the length of perpendicular ON from origin on this plane, then

$$p=\frac{q}{|\mathbf{n}|}=\frac{[\mathbf{abc}]}{n}, \quad \ldots \quad \ldots(2)$$

whereas $$\overrightarrow{ON}=p \cdot \hat{\mathbf{n}}=p \cdot \frac{\mathbf{n}}{n}=\frac{[\mathbf{abc}]}{n^2}(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})$$

[from (2)]

Note. Equation (1) is called non-parametric vector equation of a plane through three points, **a**, **b** and **c** and we have already found the corresponding parametric

equation of the plane in Chapter II § 9, Cor. 2 P. 82 as

$$\mathbf{r}=\mathbf{a}+s\,(\mathbf{b}-\mathbf{a})+t\,(\mathbf{c}-\mathbf{a}).$$

Multiply both sides of above scalarly by

$$\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}.$$

$$\begin{aligned} r\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}) \\ +s\,(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}) \\ +t\,(\mathbf{c}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}). \end{aligned}$$

Now we know that scalar triple product is zero when two of the vectors are equal. **[Cor. 2 P. 132]**

$\therefore$ coefficient of s is

$$\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})-\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=[\mathbf{abc}]-[\mathbf{abc}]=0.$$

Similarly coefficient of t is zero and hence we get

$$\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=[\mathbf{abc}]$$

which is same as equation (1) found above.

Corresponding Cartesian form.

Let in terms of unit vectors **i**, **j**, **k**

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}, \text{ and } \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k};$$

then
$$\begin{aligned} (\mathbf{r}-\mathbf{a})&=(x-a_1)\,\mathbf{i}+(y-a_2)\,\mathbf{j}+(z-a_3)\,\mathbf{k}, \\ (\mathbf{b}-\mathbf{a})&=(b_1-a_1)\,\mathbf{i}+(b_2-a_2)\,\mathbf{j}+(b_3-a_3)\,\mathbf{k}, \\ (\mathbf{c}-\mathbf{b})&=(c_1-b_1)\,\mathbf{i}+(c_2-b_2)\,\mathbf{j}+(c_3-b_3)\,\mathbf{k}. \end{aligned}$$

Since $\mathbf{r}-\mathbf{a}$, $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{b}$ are coplanar, we have

$$\begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1-b_1 & c_2-b_2 & c_3-b_3 \end{vmatrix}=0$$

as the required equation of plane.

II. Plane through a given point and parallel to two given lines.

Let the plane pass through the point **a** and parallel to lines which are parallel to **b** and **c**.

Since the plane is parallel to **b** and **c** is therefore perpendicular to $\mathbf{b}\times\mathbf{c}$, hence the required plane is one through **a** and perpendicular to $\mathbf{b}\times\mathbf{c}$ and its equation therefore is $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c})=0.$

or $$\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=[\mathbf{abc}] \qquad \ldots \quad \ldots \quad \ldots(2)$$

Note :—The corresponding parametric equation of the plane is $\mathbf{r}=\mathbf{a}+s\,\mathbf{b}+t\,\mathbf{c}$ **[Cor. 1 P. 81]** and on multiplying both sides scalarly by $\mathbf{b}\times\mathbf{c}$, we get the form (2).

III. The plane containing a given line and parallel to another line or perpendicular to a given plane $\mathbf{r}\cdot\mathbf{c}=q$.

Let the plane contain the line $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ and is parallel to **c** which means perpendicular to plane $\mathbf{r}\cdot\mathbf{c}=q$.

Thus the plane contains the point **a** and is parallel to both **b** and **c** and therefore perpendicular to $\mathbf{b}\times\mathbf{c}$ and hence its equation is $$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c})=0$$
$$\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=[\mathbf{abc}].$$

IV. The plane through two given points and parallel to a given line. (Pb. 60)

Let the plane pass through two points **a** and **b** and given line be parallel to **c**. Thus the plane is one through the point **a** and parallel to $\mathbf{b}-\mathbf{a}$ and **c** and therefore perpendicular to $(\mathbf{b}-\mathbf{a})\times\mathbf{c}$. Hence its equation is

$$(\mathbf{r}-\mathbf{a})\cdot\{(\mathbf{b}-\mathbf{a})\times\mathbf{c}\}$$

or $$\mathbf{r}\cdot\{(\mathbf{b}-\mathbf{a})\times\mathbf{c}\}=\mathbf{a}\cdot\{\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=[\mathbf{abc}].$$

V. The plane containing a given line and a given point.

Let the line be $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ and given point **c** so that the plane is through **a** and **c** and parallel to **b** *i. e.* through **a** and parallel to $\mathbf{a}-\mathbf{c}$ and **b** or perpendicular to $(\mathbf{a}-\mathbf{c})\times\mathbf{b}$.

Hence its equation is $(\mathbf{r}-\mathbf{a})\cdot[(\mathbf{a}-\mathbf{c})\times\mathbf{b}]=0$

or $$\mathbf{r}\cdot[(\mathbf{a}-\mathbf{c})\times\mathbf{b}]=\mathbf{a}\cdot[\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{b}]$$
$$=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=[\mathbf{abc}].$$

§ 3. Angle between two planes.

Let the two planes be $\mathbf{r} \cdot \mathbf{n}_1 = q_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ and the moduli of $\mathbf{n}_1$ and $\mathbf{n}_2$ be n_1 and n_2 respectively.

Now angle between two planes is equal to the angle between the normals to the planes. If θ be the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, then $\mathbf{n}_1 \cdot \mathbf{n}_2 = n_1 n_2 \cos\theta$.

$$\therefore \quad \theta = \cos^{-1} \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{n_1 n_2}.$$

Angle between a line and a plane.

Let the line be parallel to $\mathbf{b}$ and the plane be $\mathbf{r} \cdot \mathbf{n} = q$.

Now angle between a line and a plane is complement of the angle between the line and the normal to the plane.

If θ be the angle between the line and plane and ϕ be the angle between normal and line, then $\phi = 90 - \theta$.

Now $$\mathbf{n} \cdot \mathbf{b} = nb \cos\phi.$$

$$\therefore \quad \cos\phi = \frac{\mathbf{n} \cdot \mathbf{b}}{nb} = \cos(90 - \theta) = \sin\theta.$$

$$\therefore \quad \theta = \sin^{-1} \frac{\mathbf{n} \cdot \mathbf{b}}{nb}.$$

§ 4. Intercepts on axes of coordinates (rectangular).

Let the equation of the plane be $\mathbf{r} \cdot \mathbf{n} = q$.

Let the unit vectors along the axes be denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively. If x be the intercept made by the plane on the axis of x, then the point $x\mathbf{i}$ lies on the plane.

$$\therefore \quad x\mathbf{i} \cdot \mathbf{n} = q \qquad \text{or} \qquad x = \frac{q}{\mathbf{i} \cdot \mathbf{n}} \text{ respectively.}$$

Similarly the intercepts on the y and z-axis are

$$\frac{q}{\mathbf{j} \cdot \mathbf{n}} \text{ and } \frac{q}{\mathbf{k} \cdot \mathbf{n}}.$$

Ex. 2. *Prove that sum of the reciprocals of the squares of the intercepts on rectangular axes made by a fixed plane is same for all systems of rectangular axes with a given origin.*

Let the plane be $\mathbf{r} \cdot \mathbf{n} = q$.

If x, y and z are the intercepts on the axes, then

$$x = \frac{q}{\mathbf{i} \cdot \mathbf{n}}, \quad y = \frac{q}{\mathbf{j} \cdot \mathbf{n}}, \quad z = \frac{q}{\mathbf{k} \cdot \mathbf{n}} \qquad \ldots \quad \ldots (1)$$

If the normal makes angles θ_1, θ_2, θ_3 with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, then $\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3 = 1$, *i.e.* $l^2 + m^2 + n^2 = 1$.

Also $\mathbf{i} \cdot \mathbf{n} = 1 \cdot n \cos \theta_1$, $\mathbf{j} \cdot \mathbf{n} = 1 \cdot n \cos \theta_2$

and $$\mathbf{k} \cdot \mathbf{n} = 1 \cdot n \cos \theta_3 \qquad \ldots \quad \ldots (2)$$

$$\therefore \quad \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{n^2}{q^2} (\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3) = \frac{n^2}{q^2}.$$

[from (1) and (2)],

i.e. constant because the plane is fixed.

§ 5. Perpendicular distance of a point from a plane. (Agra 41)

Let the equation of the plane be $\mathbf{r} \cdot \mathbf{n} = q$, so that perpendicular from the origin O on it, *i.e.* $ON = p = q/n$.

We have to find the perpendicular distance of a point A, *i.e.* $\mathbf{a}$ from the given plane.

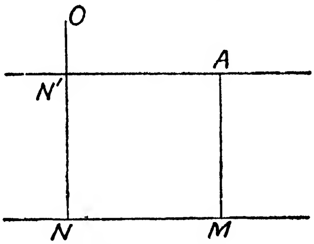


Fig. No. 83

Now consider a plane through the point A and parallel to the given plane whose equation is

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0 \quad \text{or} \quad \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

$ON' = p' =$ length of perpendicular from origin on this plane is $\dfrac{\mathbf{a} \cdot \mathbf{n}}{n}$.

$$\therefore \quad AM = ON - ON' = p - p' = \frac{q}{n} - \frac{\mathbf{a} \cdot \mathbf{n}}{n} = \frac{q - \mathbf{a} \cdot \mathbf{n}}{n}.$$

Also $$\overrightarrow{AM} = \frac{q - \mathbf{a} \cdot \mathbf{n}}{n} \hat{\mathbf{n}} = \frac{q - \mathbf{a} \cdot \mathbf{n}}{n^2} \, n \, \hat{\mathbf{n}}.$$

Whenever we have to specify a vector, we multiply its magnitude with a unit vector in that direction

or
$$\vec{AM}=\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}\,\mathbf{n}.$$

The above value of AM is positive for all those points which lie on the same side of the plane as the origin. For points on the opposite side it will be $-$ive.

Alternative Method.

Let AM be perpendicular from $\mathbf{a}$ to the plane $\mathbf{r}\cdot\mathbf{n}=q$; then its equation is $r=\mathbf{a}+t\mathbf{n}$....(1) as AM passes through $\mathbf{a}$ and being perpendicular to plane is parallel to normal $\mathbf{n}$.

M is the intersection of this line and the plane and hence we should have

$$(\mathbf{a}+t\mathbf{n})\cdot\mathbf{n}=q \quad \text{or} \quad \mathbf{a}\cdot\mathbf{n}+tn^2=q, \qquad [\because \mathbf{n}^2=n^2]$$

or
$$t=\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}.$$

Putting the value of t in (1), we get the point M as

$$\mathbf{a}+\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}\,\mathbf{n}.$$

$$\therefore \quad \vec{AM}=\vec{OM}-\vec{OA}=\mathbf{a}+\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}\,\mathbf{n}-\mathbf{a}$$

$$=\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}\cdot\mathbf{n}.$$

$$AM=|\vec{AM}|=\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}\cdot|\mathbf{n}|=\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}\cdot n$$

$$=\frac{q-\mathbf{a}\cdot\mathbf{n}}{n}.$$

Ex. 3. *Show by vector method that the points (1, 1, 1) and (2, 1, $-$4) are on opposite sides of the plane $3x+4y+5z=9$. Find the perpendicular distance of the former from the plane and the vector through it perpendicular to the plane.*

In terms of unit vectors **i**, **j**, **k**,

$$\mathbf{A}=\mathbf{i}+\mathbf{j}+\mathbf{k},\ \mathbf{B}=2\mathbf{i}+\mathbf{j}-4\mathbf{k},$$

and the given plane is $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(3\mathbf{i}+4\mathbf{j}+5\mathbf{k})=9$ so that $\mathbf{n}=3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ and $n=\sqrt{(3^2+4^2+5^2)}=5\sqrt{2}$.

Now perpendicular distance of a point from a plane is

$$\frac{q-\mathbf{a}\cdot\mathbf{n}}{n}.$$

∴ perpendicular distance from A, *i.e.* $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is

$$\frac{9-(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(3\mathbf{i}+4\mathbf{j}+5\mathbf{k})}{5\sqrt{2}}=\frac{9-(3+4+5)}{5\sqrt{2}}=\frac{-3}{5\sqrt{2}}.$$

Perpendicular distance from B, *i.e.* $2\mathbf{i}+\mathbf{j}-4\mathbf{k}$ is

$$\frac{9-(2\mathbf{i}+\mathbf{j}-4\mathbf{k})\cdot(3\mathbf{i}+4\mathbf{j}+5\mathbf{k})}{5\sqrt{2}}=\frac{9-(6+4-20)}{5\sqrt{2}}=\frac{19}{5\sqrt{2}}.$$

Again since $q-\mathbf{a}\cdot\mathbf{n}=-3$ and $q-\mathbf{b}\cdot\mathbf{n}=19$, *i.e.* they are of opposite signs, therefore the two points are on opposite sides of the plane. **[Cor. § 1 P. 160]**

Also the vector through A perpendicular to plane is

$$\frac{q-\mathbf{a}\cdot\mathbf{n}}{n^2}\cdot\mathbf{n}=\frac{-3}{(5\sqrt{2})^2}\cdot(3\mathbf{i}+4\mathbf{j}+5\mathbf{k}). \qquad [\S\ 5]$$

(b) *Show that the points* $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ *and* $3(\mathbf{i}+\mathbf{j}+\mathbf{k})$ *are equidistant from the plane* $\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j}-7\mathbf{k})+9=0$ *and lie on opposite sids of it.* **(Agra 59)**

Cor. Distance of a point from a plane measured in a given direction.

Here we have to find the distance of the point A from the plane but measured in a given direction say of the unit vector $\hat{\mathbf{b}}$.

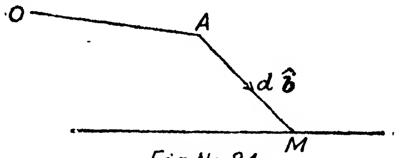


Fig No. 84

Let a line through A parallel to unit vector $\hat{\mathbf{b}}$ meet the plane

in M and let d be the length AM so that $\overrightarrow{AM}=d\ \hat{\mathbf{b}}$.

The position vector of M, *i. e.* $\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\mathbf{a}+d\ \hat{\mathbf{b}}$ lies on the plane $\mathbf{r}\cdot\mathbf{n}=q$.

$$\therefore\quad (\mathbf{a}+d\hat{\mathbf{b}})\cdot\mathbf{n}=q$$

or
$$d=\frac{q-\mathbf{a}\cdot\mathbf{n}}{\hat{\mathbf{b}}\cdot\mathbf{n}}$$

which is the required distance.

Ex. 4. *Find the distance of the point (1, 2, 3) from the plane $x+y+z=5$ measured parallel to the line*

$$\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}.$$

The given point (1, 2, 3) is $\mathbf{i}+2\mathbf{j}+3\mathbf{k}=\mathbf{a}$ say.

The given plane is $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=5$ so that $\mathbf{n}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $q=5$.

The direction ratios of the line are 2, 3, -6 and hence a vector in this direction is $2\mathbf{i}+3\mathbf{j}-6\mathbf{k}=\mathbf{b}$.

$\therefore$ a unit vector in this direction is

$$\hat{\mathbf{b}}=\frac{2\mathbf{i}+3\mathbf{j}-6\mathbf{k}}{\sqrt{(2^2+3^2+6^2)}}=\tfrac{1}{7}(2\mathbf{i}+3\mathbf{j}-6\mathbf{k}).$$

Hence the distance of A from the plane is

$$\frac{q-\mathbf{a}\cdot\mathbf{n}}{\hat{\mathbf{b}}\cdot\mathbf{n}}=\frac{5-(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\frac{1}{7}(2\mathbf{i}+3\mathbf{j}-6\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}$$

$$=\frac{5-(1+2+3)}{\frac{1}{7}(2+3-6)}=7.$$

§ 6. Planes bisecting the angles between the given planes.

Let the equations of the two planes be

$$\mathbf{r}\cdot\mathbf{n}_1=q_1 \text{ and } \mathbf{r}\cdot\mathbf{n}_2=q_2.$$

Now if $\mathbf{r}$ be any point on the plane bisecting the angle between the given planes, then the perpendicular-distance of $\mathbf{r}$ from both the planes should be equal.

or $$\frac{q_1-\mathbf{r}\cdot\mathbf{n}_1}{n_1}=\pm\frac{q_2-\mathbf{r}\cdot\mathbf{n}_2}{n_2}$$

or $$\frac{q_1}{n_1}-\mathbf{r}\cdot\hat{\mathbf{n}}_1=\pm\left(\frac{q_2}{n_2}-\mathbf{r}\cdot\hat{\mathbf{n}}_2\right), \quad \because \quad \frac{\mathbf{n}_1}{n}=\hat{\mathbf{n}}_1$$

or $$\mathbf{r}\cdot(\hat{\mathbf{n}}_1+\hat{\mathbf{n}}_2)=\left(\frac{q_1}{n_1}+\frac{q_2}{n_2}\right)$$

and $$\mathbf{r}\cdot(\hat{\mathbf{n}}_1-\hat{\mathbf{n}}_2)=\left(\frac{q_1}{n_1}-\frac{q_2}{n_2}\right).$$

Above are the required equations of the planes.

§ 7. Plane through the intersection of two given planes.

Let the given planes be $\mathbf{r}\cdot\mathbf{n}_1=q_1$ and $\mathbf{r}\cdot\mathbf{n}_2=q_2$.

Consider the equation $(\mathbf{r}\cdot\mathbf{n}_1-q_1)-\lambda\,(\mathbf{r}\cdot\mathbf{n}_2-q_2)=0\ldots(1)$ where λ is any constant.

Now all those points which satisfy both the given planes also satisfy the equation (1) for all values of λ and as such it is satisfied by all the points on the line of intersection of the given planes. Rewriting (1) in the form $\mathbf{r}\cdot(\mathbf{n}_1-\lambda\mathbf{n}_2)=q_1-\lambda q_2$, we get the required equation of the plane through the line of intersection of the given planes. The value of λ is found by an additional given condition of the question. For example, if the required plane passes through a given point $\mathbf{a}$, then

$$\mathbf{a}\cdot(\mathbf{n}_1-\lambda\mathbf{n}_2)=q_1-\lambda q_2$$

or $$\lambda=\frac{\mathbf{a}\cdot\mathbf{n}_1-q_1}{\mathbf{a}\cdot\mathbf{n}_2-q_2}.$$

In particular, if the required plane passes theough origin, then putting $\mathbf{a}=0$, we get $\lambda=q_1/q_2$ and hence the

equation of the plane is $q_2(\mathbf{r}\cdot\mathbf{n}_1-q_1)-q_1(\mathbf{r}\cdot\mathbf{n}_2-q_2)=0$ or $\mathbf{r}\cdot(q_2\mathbf{n}_1-q_1\mathbf{n}_2)=0$.

§ 8. Line of intersection of two planes.

(Agra 51, 55, 61) [Read carefully]

Let te equations of the two planes be

$$\mathbf{r}\cdot\mathbf{n}_1=q_1 \quad\text{and}\quad \mathbf{r}\cdot\mathbf{n}_2=q_2.$$

The line of intersection being common to both the planes is therefore perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$ the normals of the two given planes. But $\mathbf{n}_1\times\mathbf{n}_2$ represents a vector perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$ and hence we conclude that the line is parallal to $\mathbf{n}_1\times\mathbf{n}_2$. In order to determine its equation completely, we must know a point on it. Now if N be the foot of the perpendicular from origin O on the line, then clearly ON is parallel to the plane of $\mathbf{n}_1$ and $\mathbf{n}_2$. If we denote the position vector of N by $\mathbf{a}$, then $\mathbf{a}=l_1\mathbf{n}_1+l_2\mathbf{n}_2$, where l_1 and l_2 are any constants.

Now N lies on both the planes. $\therefore$ $l_1\mathbf{n}_1+l_2\mathbf{n}_2$ should satisfy both the planes and this will give us the values of l_1 and l_2.

$$\therefore\quad (l_1\mathbf{n}_1+l_2\mathbf{n}_2)\cdot\mathbf{n}_1=q_1 \quad\text{or}\quad l_1\mathbf{n}_1^2+l_2\mathbf{n}_1\cdot\mathbf{n}_2=q_1 \qquad \ldots(1)$$

$$(l_1\mathbf{n}_1+l_2\mathbf{n}_2)\cdot\mathbf{n}_2=q \quad\text{or}\quad l_1\mathbf{n}_1\cdot\mathbf{n}_2+l_2\mathbf{n}_2^2=q_2 \qquad \ldots(2)$$

Multiplying (1) by $\mathbf{n}_2^2$ and (2) by $\mathbf{n}_1\cdot\mathbf{n}_2$ and subtracting, we get $l_1[\mathbf{n}_1^2\mathbf{n}_2^2-(\mathbf{n}_1\cdot\mathbf{n}_2)^2]=q_1\mathbf{n}_2^2-q_2\mathbf{n}_1\cdot\mathbf{n}_2$.

$$\therefore\quad l_1=\frac{q_1n_2^2-q_2\mathbf{n}_1\cdot\mathbf{n}_2}{n_1^2n_2^2-(\mathbf{n}_1\cdot\mathbf{n}_2)^2},\quad \because\ \mathbf{n}_1^2=n_1^2 \text{ and } \mathbf{n}_2^2=n_2^2.$$

Similarly, $$l_2=\frac{q_2n_1^2-q_1\mathbf{n}_1\cdot\mathbf{n}_2}{n_1^2n_2^2-(\mathbf{n}_1\cdot\mathbf{n}_2)^2}.$$

Having found l_1 and l_2, we know the position vector of point N on the line which is parallel to $\mathbf{n}_1\times\mathbf{n}_2$ and hence its equation is

$$r=l_1\mathbf{n}_1+l_2\mathbf{n}_2+t\mathbf{n}_1\times\mathbf{n}_2.$$

Note. Above is called **parametric** equation of the line of intersection.

We can however find the **non-parametric** equation of the line as follows.

Equation of a line through a given point and parallel to a given line.

Let $\mathbf{r}$ be any point on the line which passes through a given point $\mathbf{a}$ and which is parallel to $\mathbf{b}$. $\therefore$ $\mathbf{r}-\mathbf{a}$ and $\mathbf{b}$ are two parallels and we know that cross product of two parallel vectors is zero.

$$\therefore \quad (\mathbf{r}-\mathbf{a})\times\mathbf{b}=0 \quad \text{or} \quad \mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$$

is the required non-parametric equation of the line.

In case the line passes through origin, then putting $\mathbf{a}=0$, we get its equation as $\mathbf{r}\times\mathbf{b}=0$.

In case the line is perpendicular to c and d, then clearly it is parallel to $\mathbf{c}\times\mathbf{d}$ and hence replacing $\mathbf{b}$ by $\mathbf{c}\times\mathbf{d}$ we get its equation as $(\mathbf{r}-\mathbf{a})\times(\mathbf{c}\times\mathbf{d})=0$.

Hence the non-parametric vector equation of the line of intersection of two planes is

$$(\mathbf{r}-\mathbf{a})\times(\mathbf{n}_1\times\mathbf{n}_2)=0 \qquad \ldots \qquad \ldots(2)$$

where $\mathbf{a}=l_1\mathbf{n}_1+l_2\mathbf{n}_2$ and l_1, l_2 have values found above.

It may be observed here as in Note 162 that the non-parametric form of the equation of line can be easily deduced from the corresponding parametric form

$$\mathbf{r}=\mathbf{a}+t\,\mathbf{b}. \qquad \text{[§ 6 P. 47]}$$

Multiply both sides vectorially by $\mathbf{b}$.

$$\therefore \quad \mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}+t\,\mathbf{b}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}.$$

$\therefore$ $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$ is the required equation.

Ex. 5. (a) *Find the equation of the line of intersection of the planes* $\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+\mathbf{k})=1$ *and* $\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}-2\mathbf{k})=2$. **(Agra 45)**

The line of intersection is clearly parallel to $\mathbf{n}_1\times\mathbf{n}_2$,

i.e. $$(3\mathbf{i}-\mathbf{j}+\mathbf{k})\times(\mathbf{i}+4\mathbf{j}-2\mathbf{k}),$$

i.e. $$(2-4)\,\mathbf{i}+[1-(-6)]\,\mathbf{j}+[12-(-1)]\,\mathbf{k}$$

or $$-2\mathbf{i}+7\mathbf{j}+13\mathbf{k}.$$

In order to find the line we must know a point on it.

Let $\mathbf{a}$ be the foot of the perpendicular from origin on it; then since it is expressible as a linear combination of $\mathbf{n}_1$ and $\mathbf{n}_2$, let $\mathbf{a}=l\,(3\mathbf{i}-\mathbf{j}+\mathbf{k})+l_2\,(\mathbf{i}+4\mathbf{j}-2\mathbf{k})$.

Now $\mathbf{a}$ lies on both the planes.

$$\therefore\quad [l_1\,(3\mathbf{i}-\mathbf{j}+\mathbf{k})+l_2\,(\mathbf{i}+4\mathbf{j}-2\mathbf{k})]\cdot(3\mathbf{i}-\mathbf{j}+\mathbf{k})=1$$

and $$[l_1\,(3\mathbf{i}-\mathbf{j}+\mathbf{k})+l_2\,(\mathbf{i}+4\mathbf{j}-2\mathbf{k})]\cdot(\mathbf{i}+4\mathbf{j}-2\mathbf{k})=2.$$

$$\therefore\quad 11l_1-3l_2=1 \quad\text{and}\quad -3l_1+21l_2=2.$$

Solving the above two, we get

$$l_1=\frac{27}{222}\quad\text{and}\quad l_2=\frac{25}{222}.$$

Having found the point and direction, the equation of the line is given by $\mathbf{r}=\mathbf{a}+t\,\mathbf{b}$,

i.e. $$\mathbf{r}=\frac{26}{222}\,(3\mathbf{i}-\mathbf{j}+\mathbf{k})+\frac{25}{222}\,(\mathbf{i}+4\mathbf{j}-2\mathbf{k})+t\,(2\mathbf{i}+7\mathbf{j}+13\mathbf{k})$$

or $$\mathbf{r}=\tfrac{1}{222}\,(106\mathbf{i}+73\mathbf{j}-23\mathbf{k})+t\,(-2\mathbf{i}+7\mathbf{j}+13\mathbf{k}).$$

Above is the parametric equation of the line.

Again in order to find the non-parametric equation of the line,

$$(\mathbf{r}-\mathbf{a})\times(\mathbf{n}_1\times\mathbf{n}_2)=0$$ **[Bottom P. 167]**

or $$\mathbf{r}\times(-2\mathbf{i}+7\mathbf{j}+13\mathbf{k})=5\mathbf{i}\times 6\mathbf{j}+4\mathbf{k}.$$

Ex. 5. (b) *Prove that the plane through the point $(-1,-1,-1)$ and containing the line of the planes*

$$\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}-\mathbf{k})=0 \text{ and } \mathbf{r}\cdot(\mathbf{j}+2\mathbf{k})=0$$

is $$\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-3\mathbf{k})=0.$$

§ 9. Perpendicular distance of a point from a given line. **(Agra 49)**

Let the given line be $\mathbf{r}=\mathbf{a}+t\hat{\mathbf{b}}$ passing through the point A and parallel to unit vector $\hat{\mathbf{b}}$. We are to find the perpendicular distance of any point C whose position vector is $\mathbf{c}$ from this

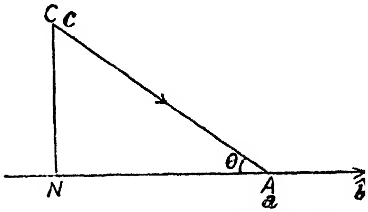


Fig No 85

line *i.e.* CN. Clearly $\vec{CA}=\vec{OA}-\vec{OC}=\mathbf{a}-\mathbf{c}$ and $CA^2=(\mathbf{a}-\mathbf{c})^2$...(1) because square of vector is square of its module.

Also NA is projection of CA along unit vector $\hat{\mathbf{b}}$ and it is therefore equal to $CA \cos\theta=1 \cdot CA \cdot \cos\theta$.

$$=\hat{\mathbf{b}}\cdot(\mathbf{a}-\mathbf{c});\quad \therefore\quad NA^2=[\hat{\mathbf{b}}\cdot(\mathbf{a}-\mathbf{c})]^2 \quad \ldots \quad \ldots(2)$$

$$\therefore\quad CN^2=CA^2-NA^2=(\mathbf{a}-\mathbf{c})^2-[\hat{\mathbf{b}}\cdot(\mathbf{a}-\mathbf{c})]^2$$

from (1) and (2).

Also $\vec{CN}=\vec{CA}+\vec{AN}=\vec{CA}-\vec{NA}$

$$=(\mathbf{a}-\mathbf{c})-[\hat{\mathbf{b}}\cdot(\mathbf{a}-\mathbf{c})]\,\hat{\mathbf{b}}$$

because $NA=\hat{\mathbf{b}}\cdot(\mathbf{a}-\mathbf{c})$ and it being in the direction of $\hat{\mathbf{b}}$,

$$\therefore\quad \vec{NA}=[\hat{\mathbf{b}}\cdot(\mathbf{a}-\mathbf{c})]\,\hat{\mathbf{b}}.$$

In case $\mathbf{b}$ be not a unit vector then we shall replace $\hat{\mathbf{b}}$ by $\mathbf{b}/b$ where b is the module of $\mathbf{b}$.

Ex. 6. *Find the perpendicular distance of a corner of a unit cube from a diagonal not passing through it.* **(Agra 33, 41, 56)**

Since the cube is a unit cube, let the vectors determined

by conterminous edges OA, OB and OC be **i**, **j** and **k** respectively so that the diagonal $\vec{OP}$ is $\mathbf{b}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ which passes through origin.

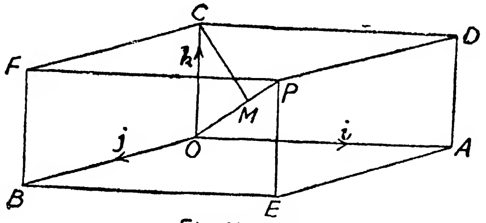


Fig. No. 86

A unit vector in this direction is $\dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$.

If CM be perpendicular from C *i. e.* **k** on OP, then

OM=projection of OC on OP

$$=\mathbf{k}\,.\text{ unit vector along }\vec{OP}$$

$$=\mathbf{k}\,.\,\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}=\frac{1}{\sqrt{3}}.$$

Also $\vec{OC}=\mathbf{k}$; $\therefore\quad OC=|\vec{OC}|=1.$

$\therefore\quad CM^2=OC^2-OM^2=1-\frac{1}{3}=\frac{2}{3};\quad\therefore\quad CM=\sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{6}.$

§ 10. To find the condition that any two given lines may be coplanar i. e. they may intersect.

Let the given lines be

$\mathbf{r}=\mathbf{a}_1+t\mathbf{b}_1$

and $\quad\mathbf{r}=\mathbf{a}_2+s\mathbf{b}_2$

i. e. passing through $\mathbf{a}_1$, $\mathbf{a}_2$ and parallel to $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively. In case they intersect, then their common plane should be

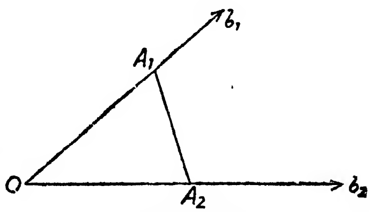


Fig. No. 87

parallel to $\mathbf{a}_1-\mathbf{a}_2$, $\mathbf{b}_1$ and $\mathbf{b}_2$ *i. e.* these three vectors should be coplanar, the condition for which is that

$$(\mathbf{a}_1-\mathbf{a}_2)\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0 \quad \textbf{[Cor. 1 P. 137]}$$

or
$$\mathbf{a}_1\cdot(\mathbf{b}_1\times\mathbf{b}_2)=\mathbf{a}_2\cdot(\mathbf{b}_1\times\mathbf{b}_2)$$

or
$$[\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2]=[\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2] \qquad \ldots \quad \ldots \quad \ldots(1)$$

is the required condition.

The plane through the coplanar lines.

In case the given lines intersect, then condition (1) holds good and in order to find the equation of the plane through them, we write the equation of the plane through the point $\mathbf{a}_1$ and parallel to $\mathbf{b}_1$ and $\mathbf{b}_2$. Since plane is parallel to $\mathbf{b}_1$ and $\mathbf{b}_2$ it is perpendicular to $\mathbf{b}_1\times\mathbf{b}_2$ which is perpendicular to $\mathbf{r}-\mathbf{a}_1$, any line in the plane.

$\because$ dot product of two perpendicular vectors is zero,

$$\therefore \quad (\mathbf{r}-\mathbf{a}_1)\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0$$

or
$$\mathbf{r}\cdot(\mathbf{b}_1\times\mathbf{b}_2)=\mathbf{a}_1\cdot(\mathbf{b}_1\times\mathbf{b}_2)$$

or
$$[\mathbf{r}\mathbf{b}_1\mathbf{b}_2]=[\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2] \qquad \ldots \quad \ldots \quad \ldots(2)$$

Above is the required equation of the plane and it will pass through the point $\mathbf{a}_2$ if $[\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2]=[\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2]$ which we know is true by virtue of (2). Hence (2) represents the required plane.

Note :—By considering the plane through $\mathbf{a}_2$ we could also write its equation as

$$[\mathbf{r}\mathbf{b}_1\mathbf{b}_2]=[\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2].$$

Corresponding cartesian form.

[See author's Solid Geometry P. 125]

Let in terms of any three non-coplanar unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along the axes,

$$\mathbf{a}_1=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k} \text{ and } \mathbf{b}_1=l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k},$$
$$\mathbf{a}_2=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k} \text{ and } \mathbf{b}_2=l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k}$$

so that the two lines are $\mathbf{r}=\mathbf{a}_1+t\mathbf{b}_1$ and $\mathbf{r}=\mathbf{a}_2+t\mathbf{b}_2$.

Taking the corresponding cartesian form where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, we get their equations as (Cor. 3 P. 48)

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}.$$

The condition for coplanar lines is

$$(\mathbf{a}_1-\mathbf{a}_2)\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0$$

where $\quad \mathbf{a}_1-\mathbf{a}_2=(x_1-x_2)\,\mathbf{i}+(y_1-y_2)\,\mathbf{j}+(z_1-z_2)\,\mathbf{k}.$

The above condition means that

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}[\mathbf{ijk}]=0 \qquad \text{[Cor. 4 P. 138]}$$

$[\mathbf{ijk}]\neq 0$ but equal to 1, $\therefore$ the required condition is

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Also the equation of the plane containing them is

$$(\mathbf{r}-\mathbf{a}_1)\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0 \quad \text{or} \quad (\mathbf{r}-\mathbf{a}_2)\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0$$

or
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

or
$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

§ 11. Shortest distance between two non-intersecting lines. **(Agra 37, 39, 42, 52)**

Let the equations of the lines be

$$\mathbf{r}=\mathbf{a}_1+t\mathbf{b}_1 \text{ and } \mathbf{r}=\mathbf{a}_2+s\mathbf{b}_2.$$

Let P_1P_2 be the shortest distance between the given lines. Since P_1P_2 is perpendicular to both $\mathbf{b}_1$ and $\mathbf{b}_2$ it is parallel to $\mathbf{b}_1\times\mathbf{b}_2=\mathbf{n}$ say whose module is n.

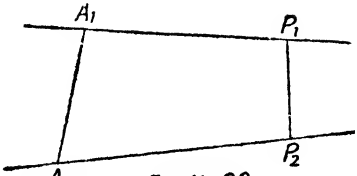


Fig. No 88

Let A_1 *i.e.* $\mathbf{a}_1$ and A_2 *i.e.* $\mathbf{a}_2$ be any two points on the given lines respectively, then shortest distance is the projection of A_1A_2 on P_1P_2, *i.e.* projection of $\mathbf{a}_1-\mathbf{a}_2$ on $\mathbf{n}$

$$=\frac{(\mathbf{a}_1-\mathbf{a}_2)\cdot\mathbf{n}}{n}=\frac{(\mathbf{a}_1-\mathbf{a}_2)\cdot(\mathbf{b}_1\times\mathbf{b}_2)}{|\mathbf{b}_1\times\mathbf{b}_2|}.$$

Thus in order to find the S. D. between two non-intersecting lines parallel respectively to $\mathbf{b}_1$ and $\mathbf{b}_2$, you should find a point on each of the lines; then find the projection of the line joining these points on $\mathbf{b}_1\times\mathbf{b}_2$.

Equation of the shortest distance.

The equation of the shortest distance is the line of intersection of the planes through the given lines and the shortest distance.

The plane containing $\mathbf{r}=\mathbf{a}_1+t\mathbf{b}_1$ and S. D. which is parallel to $\mathbf{b}_1\times\mathbf{b}_2$ is

$$(\mathbf{r}-\mathbf{a}_1)\cdot[\mathbf{b}_1\times(\mathbf{b}_1\times\mathbf{b}_2)]=0. \qquad [\S\ 2\ \text{III P. } 164]$$

The plane containing $\mathbf{r}=\mathbf{a}_2+s\mathbf{b}_2$ and S. D. is

$$(\mathbf{r}-\mathbf{a}_2)\cdot[\mathbf{b}_2\times(\mathbf{b}_1\times\mathbf{b}_2)]=0. \qquad [\S\ 2\ \text{III P. } 164].$$

Note. In case the line be coplanar, then S. D. between them is zero and as such $(\mathbf{a}_1-\mathbf{a}_2)\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0$
or $\mathbf{a}_1\cdot(\mathbf{b}_1\times\mathbf{b}_2)=\mathbf{a}_2\cdot(\mathbf{b}_1\times\mathbf{b}_2)=0$ or $[\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2]=[\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2]$ as found in § 10.

Corresponding cartesian form.

Resolving in terms of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as in § 10,

S. D. is $\dfrac{(\mathbf{a}_1-\mathbf{a}_2)\cdot(\mathbf{b}_1\times\mathbf{b}_2)}{|\,\mathbf{b}_1\times\mathbf{b}_2\,|}=\dfrac{[\mathbf{a}_1-\mathbf{a}_2\ \mathbf{b}_1\ \mathbf{b}_2]}{|\,\mathbf{b}_1\times\mathbf{b}_2\,|}$

or
$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div |\,\mathbf{b}_1\times\mathbf{b}_2\,| \qquad \ldots \quad \ldots(1)$$

$$\begin{aligned}(\mathbf{b}_1\times\mathbf{b}_2)^2&=b_1{}^2b_2{}^2\sin^2\theta=b_1{}^2b_2{}^2\cos^2\theta\\ &=b_1{}^2b_2{}^2-(\mathbf{b}_1\cdot\mathbf{b}_2)^2\\ &=(l_1{}^2+m_1{}^2+n_1{}^2)(l_2{}^2+m_2{}^2+n_2{}^2)-(l_1l_2+m_1m_2+n_1n_2)^2\\ &=\Sigma\ (m_1n_2-m_2n_1)^2.\end{aligned}$$

$$\therefore \quad |\,\mathbf{b}_1\times\mathbf{b}_2\,|=\surd[\Sigma(m_1n_2-m_2n_1)^2].$$

Alternative. $\mathbf{b}_1\times\mathbf{b}_2=(l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k})\times(l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k})$

$$=(m_1n_2-m_2n_1)\ \mathbf{i}+(n_1l_2-n_2l_1)\ \mathbf{j}$$
$$+(l_1m_2-l_2m_1)\ \mathbf{k}.$$

$$\therefore \quad |\,\mathbf{b}_1\times\mathbf{b}_2\,|=\surd[\Sigma m_1n_2-m_2n_1)^2].$$

On putting for $|\,\mathbf{b}_1\times\mathbf{b}_2\,|$ in (1), we get S. D.

The equation of the S. D. is the line of intersection of planes.

$$(\mathbf{r}-\mathbf{a}_1)\cdot[\mathbf{b}_1\times(\mathbf{b}_1\times\mathbf{b}_2)=0$$

i.e. $[\mathbf{r}-\mathbf{a}_1\quad \mathbf{b}_1\quad \mathbf{b}_1\times\mathbf{b}_2]=0$

and $[\mathbf{r}-\mathbf{a}_2\quad \mathbf{b}_2\quad \mathbf{b}_1\times\mathbf{b}_2]=0,$

i.e.
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_2m_1 \end{vmatrix}=0$$

and
$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_1m_1 \end{vmatrix}=0.$$

Ex. 7. *The shortest distances between a diagonal of a rectangular parallelopiped whose sides are a, b, c and the edges not*

meeting it are

$$\frac{bc}{\sqrt{(b^2+c^2)}}, \frac{ca}{\sqrt{(c^2+a^2)}}, \frac{ab}{\sqrt{(a^2+b^2)}}. \quad \textbf{(Agra 33, 60)}$$

[See Author's Solid Geometry]

Let the unit vectors along OA, OB and OC be $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Now $OA=a$, $OB=b$ and $OC=c$ and therefore position vectors of A, B and C are $a\mathbf{i}$, $b\mathbf{j}$ and $c\mathbf{k}$ respectively.

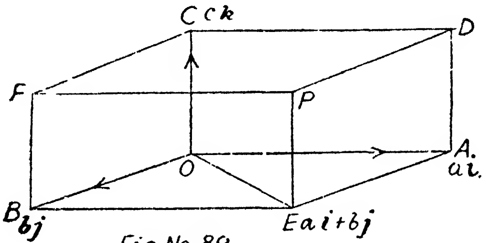


Fig. No 89

Now we have to find the shortest distance between the diagonal EC and the edge OB which does not meet it.

$$\vec{OB}=b\mathbf{j} \text{ and } \vec{EC}=\vec{OC}-\vec{OE}=c\mathbf{k}-(a\mathbf{i}+b\mathbf{j}).$$

Shortest distance is $\dfrac{(\mathbf{a}-\mathbf{a}') \cdot (\mathbf{b}\times\mathbf{b}')}{|\mathbf{b}\times\mathbf{b}'|}$ where $\mathbf{a}$ and $\mathbf{a}'$ are points on each of the lines and $\mathbf{b}$ and $\mathbf{b}'$ are the directions of the lines.

$$\mathbf{a}=0, \text{ a point } O \text{ on } OB$$

and $$\mathbf{a}'=c\mathbf{k}, \text{ a point } C \text{ on } EC.$$

Also $\mathbf{b}\times\mathbf{b}'=\vec{OB}\times\vec{EC}=b\mathbf{j}\times(c\mathbf{k}-a\mathbf{i}-b\mathbf{j})$

$$=bc\ \mathbf{j}\times\mathbf{k}-ba\ \mathbf{j}\times\mathbf{i}, \quad \because \quad \mathbf{j}\times\mathbf{j}=0$$

$$=bc\ \mathbf{i}+ab\ \mathbf{k}, \because \mathbf{j}\times\mathbf{k}=\mathbf{i} \text{ and } \mathbf{i}\times\mathbf{j}=-\mathbf{j}\times\mathbf{i}=\mathbf{k}.$$

$$\therefore \quad |\mathbf{b}\times\mathbf{b}'|=\sqrt{(b^2c^2+a^2b^2)}=b\sqrt{(a^2+c^2)}.$$

$$\therefore \quad \text{S.D.}=\frac{(0-c\mathbf{k}) \cdot (bc\ \mathbf{i}+ab\ \mathbf{k})}{b\sqrt{(a^2+c^2)}}$$

$$=\frac{-abc}{b\sqrt{(a^2+c^2)}}=\frac{ac}{\sqrt{(a^2+c^2)}}.$$

Similarly we can find the S. D. between other diagonals and the edges.

Exercises

Ex. 1. *Prove that the equation of the plane through the point (1, 2, 3) and perpendicular to each of the planes* $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=3$ *and* $\mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})=0$ *is* $\mathbf{r}\cdot(\mathbf{i}+6\mathbf{j}+5\mathbf{k})=28$.

If $\mathbf{n}$ be the normal to the required plane, it is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$, the normals of other given planes, and hence parallel to $\mathbf{b}$ and $\mathbf{c}$. If $\mathbf{a}$ be the given point, then the plane is $$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c})=0 \qquad \text{[Cor. 2 P. 159]}$$
or $$[\mathbf{rbc}]=[\mathbf{abc}].$$

Ex. 2. *Find the equation of the plane through the point (1, 2, −1) and perpendicular to the line of intersection of planes* $\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+\mathbf{k})=1$ *and* $\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}-2\mathbf{k})=2$.

The given point is $\mathbf{i}+2\mathbf{j}-\mathbf{k}=\mathbf{a}$ say and from Ex. 5 P. 173 the line of intersection is parallel to $-2\mathbf{i}+7\mathbf{j}+13\mathbf{k}=\mathbf{n}$ say and it being perpendicular to the plane its equation is given by $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$.

Ans. $\mathbf{r}\cdot(2\mathbf{i}-7\mathbf{j}+13\mathbf{k})=1$.

Ex. 3. *Find the line of intersection of the planes* $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}-\mathbf{k})=0$ *and* $\mathbf{r}\cdot(\mathbf{j}+2\mathbf{k})=0$ *and hence find the equation of the plane containing the above line and through the point (−1, −1, −1).*

The line of intersection is parallel to
$$(\mathbf{i}+3\mathbf{j}-\mathbf{k})\times(\mathbf{j}+2\mathbf{k})=7\mathbf{i}-2\mathbf{j}+\mathbf{k}$$
since both the planes pass through origin, and hence the line also passes through origin. The plane also passes through (−1, −1, −1), *i. e.* $-\mathbf{i}-\mathbf{j}-\mathbf{k}$.

Now use case V. P. 164.

Ans. $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-3\mathbf{k})=0$.

Ex. 4. *Prove that the planes* $\mathbf{r} \cdot (2\mathbf{i}+5\mathbf{j}+3\mathbf{k})=0$ *and* $\mathbf{r} \cdot (\mathbf{i}-\mathbf{j}+4\mathbf{k})=2$ *intersect in the line*

$$\mathbf{r}=-\frac{2}{67}(2\mathbf{i}+5\mathbf{j}+3\mathbf{k})+\frac{76}{6\times 67}(\mathbf{i}-\mathbf{j}+4\mathbf{k})+t\,(23\mathbf{i}-5\mathbf{j}-7\mathbf{k})$$

and hence show that the given planes and the plane

$$\mathbf{r} \cdot (7\mathbf{j}-5\mathbf{k})+4=0$$

have a common line of intersetion.

Hint. Prove that the line of intersection lies in the third plane *i. e.* a point on it satisfies the plane and it is perpendicular to the normal.

Ex. 5. *Prove that the line of intersection of the planes* $\mathbf{r} \cdot (\mathbf{i}+2\mathbf{j}+3\mathbf{k})=0$ *and* $\mathbf{r} \cdot (3\mathbf{i}+2\mathbf{j}+\mathbf{k})=0$ *is* $\mathbf{r}=t\,(\mathbf{i}-2\mathbf{j}+\mathbf{k})$. *Show that the line is equally inclined to* $\mathbf{i}$ *and* $\mathbf{k}$ *and makes an angle* $\frac{1}{2}\sec^{-1} 3$ *with* $\mathbf{j}$. **(Agra 55, 61; Utkal 53)**

Angle between line and $\mathbf{j}$ is given by

$$(\mathbf{i}-2\mathbf{j}+\mathbf{k}) \cdot \mathbf{j}=ab\cos\theta=\sqrt{(1+4+1)} \cdot 1 \cdot \cos\theta$$

or $$-2=\sqrt{6}\cos\theta\,; \quad \therefore \quad \cos\theta=-2/\sqrt{6}.$$

$$\therefore \quad \cos 2\theta=2\cos^2\theta-1=2 \cdot \tfrac{4}{6}-1=\tfrac{1}{3}\,;$$

$$\therefore \quad \sec 2\theta=3 \quad \text{or} \quad \theta=\tfrac{1}{2}\sec^{-1} 3 \text{ etc.}$$

Ex. 6. *Prove that the plane through the point* $\mathbf{a}$ *parallel to the line* $\mathbf{r}=\mathbf{b}+t\mathbf{c}$ *and perpendicular to plane* $\mathbf{r} \cdot \mathbf{n}=q$ is $[\mathbf{rnc}]=[\mathbf{anc}]$.

If $\mathbf{n}_1$ be the normal to the plane, then it is perpendicular to both $\mathbf{n}$ and $\mathbf{c}$ and hence parallel to $\mathbf{n}\times\mathbf{c}$ and therefore the required plane is $(\mathbf{r}-\mathbf{a}) \cdot (\mathbf{n}\times\mathbf{c})=0$.

Ex. 7. *Find the equation of the line through the point* $\mathbf{a}$ *parallel to the plane* $\mathbf{r} \cdot \mathbf{n}=q$ *and perpendicular to line* $\mathbf{r}=\mathbf{b}+t\mathbf{c}$.

Ans. $r=\mathbf{a}+t\mathbf{n}\times\mathbf{c}$ or $(\mathbf{r}-\mathbf{a})\times(\mathbf{n}\times\mathbf{c})=0$.

Ex. 8. *Find the equation of the plane which passes through the line of intersection of the planes* $\mathbf{r} \cdot \mathbf{n}_1=q_1$, $\mathbf{r} \cdot \mathbf{n}_2=q_2$ *and is parallel to the line of intersection of the planes*

$$\mathbf{r} \cdot \mathbf{n}_3=q_3\,, \quad \mathbf{r} \cdot \mathbf{n}_4=q_4.$$

Any plane through the line of intersection of given

planes is $(\mathbf{r}\cdot\mathbf{n}_1-q_1)+\lambda\,(\mathbf{r}\cdot\mathbf{n}_2-q_2)=0$

or $$\mathbf{r}\cdot(\mathbf{n}_1+\lambda\mathbf{n}_2)=q_1+\lambda q_2 \quad \dots \quad \dots \quad \dots(1)$$

Above plane is parallel to $\mathbf{n}_3\times\mathbf{n}_4$ *i. e.* the line of intersection of $\mathbf{r}\cdot\mathbf{n}_3=q_3$ and $\mathbf{r}\cdot\mathbf{n}_4=q_4$ and hence perpendicular to normal.

$$\therefore\quad (\mathbf{n}_1+\lambda\mathbf{n}_2)\cdot(\mathbf{n}_3\times\mathbf{n}_4)=0,$$

$$\mathbf{n}_1\cdot(\mathbf{n}_3\times\mathbf{n}_4)=-\lambda\,\{(\mathbf{n}_2\cdot(\mathbf{n}_3\times\mathbf{n}_4)\}$$

or $$-\frac{[\mathbf{n}_1\mathbf{n}_3\mathbf{n}_4]}{[\mathbf{n}_2\mathbf{n}_3\mathbf{n}_4]}=\lambda.$$

Hence the required plane is

$$(\mathbf{r}\cdot\mathbf{n}_1-q_1)\,[\mathbf{n}_2\mathbf{n}_3\mathbf{n}_4]=(\mathbf{r}\cdot\mathbf{n}_2-q_2)\,[\mathbf{n}_1\mathbf{n}_3\mathbf{n}_4].$$

Ex. 9. (a) *Find the equation of the plane which contains two parallel lines* $\mathbf{r}=\mathbf{a}+t\,\mathbf{b}$, $\mathbf{r}=\mathbf{c}+t\,\mathbf{b}$.

Clearly plane passes through $\mathbf{a}$ and is parallel to $\mathbf{a}-\mathbf{c}$ and $\mathbf{b}$. **[Case II P 158]**

Ans. $\mathbf{r}\cdot\{(\mathbf{a}-\mathbf{c})\times\mathbf{b}\}+[\mathbf{a}\mathbf{c}\mathbf{b}]=0.$

Ex. 9. (b) *Find the equation of the plane which contains the line* $\mathbf{r}=t\,\mathbf{a}$ *and is perpendicular to the plane* containing

$$\mathbf{r}=s\,\mathbf{b}\ \textit{snd}\ \mathbf{r}=k\,\mathbf{c}.$$

The plane containing $\mathbf{r}=s\,\mathbf{b}$ and $\mathbf{r}=k\,\mathbf{c}$ will be perpendicular to $\mathbf{b}\times\mathbf{c}$. The required plane being perpendicular to above plane is therefore parallel to $\mathbf{b}\times\mathbf{c}$. Also it contains $r=t\,\mathbf{a}$, *i. e.* it is parallel to $\mathbf{a}$. Hence it is perpendicular to $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. Since it passes through origin; hence it equation is $\mathbf{r}\cdot\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=0.$

Ex. 10. *Prove that the lines*

$$\mathbf{r}\times\mathbf{a}=\mathbf{b}\times\mathbf{a}\ \textit{and}\ \mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b} \qquad \textbf{(Pb. 60)}$$

intersect, and find their point of intersection.

The first line is $(\mathbf{r}-\mathbf{b})\times\mathbf{a}=0$ *i. e.* a line through $\mathbf{b}$ and parallel to $\mathbf{a}$ and its parametric equation is $\mathbf{r}=\mathbf{b}+t\,\mathbf{a}$.

Similarly the 2nd line is $\mathbf{r}=\mathbf{a}+s\,\mathbf{b}$.

They will intersect if $[\mathbf{a}-\mathbf{b},\ \mathbf{a},\ \mathbf{b}]=0$,

$$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})=0,\ \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=0,$$

which is true as scalar triple product is zero when two vectors are equal.

For their point of intersection we should have identical value of $\mathbf{r}$ for which on comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$ in their equations we have $1=s$ and $1=t$ and we get the required point as $\mathbf{a}+\mathbf{b}$.

Ex. 11. *Prove that the lines*

$$\mathbf{r}=\mathbf{a}+t\,(\mathbf{b}\times\mathbf{c}) \textit{ and } \mathbf{r}=\mathbf{b}+s\,(\mathbf{c}\times\mathbf{a})$$

will intersect if $\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}$.

Ex. 12. *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *be the position vectors of four points relative to an origin O, then interpret geometrically the equations*

$$(\mathbf{c}-\mathbf{d})\times(\mathbf{a}\times\mathbf{b})=0,\ (\mathbf{c}-\mathbf{d})\cdot(\mathbf{a}\times\mathbf{b})=0. \qquad \textbf{(Pb. 60)}$$

Clearly equation of plane OAB is $\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})=0$. Also DC is $\mathbf{c}-\mathbf{d}$. Cross product of two vectors is zero when they are parallel.

$\therefore$ CD is parallel to $\mathbf{a}\times\mathbf{b}$ which is normal to plane OAB.

Hence CD is normal to plane OAB.

Dot product is zero when vectors are perpendicular. Therefore CD is perpendicular to normal, *i. e.* it lies in the plane OAB. or is parallel to it.

Ex. 13. *Prove that the locus of a point such that the difference of the squares of its distances from two given points is constant is a plane perpendicular to the lines joining the points.* **(Agra 43)**

Ex. 14. *Find the equation of the straight lines through the point* $\mathbf{a}$ *and intersecting both the lines* $\mathbf{r}=\mathbf{c}+t\,\mathbf{d}$ *and* $\mathbf{r}=\mathbf{c}'+t'\,\mathbf{d}'$.

(Agra 46, 55, 61, Delhi 51)

Let the equation of the given line be

$$\mathbf{r}=\mathbf{a}+s\,\mathbf{b} \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

and the given lines are

$$\mathbf{r}=\mathbf{c}+t\,\mathbf{d} \quad \ldots \quad \ldots \quad \ldots \quad \ldots(2)$$

$$\mathbf{r}=\mathbf{c}'+t'\,\mathbf{d}' \quad \ldots \quad \ldots \quad \ldots \quad \ldots(3)$$

Since (1) and (2) intersect, we have $[\mathbf{b}\ \mathbf{d}\ \mathbf{c}-\mathbf{a}]=0$

Again (1) and (3) also intersect. $\therefore$ $[\mathbf{b}\ \mathbf{d}'\ \mathbf{c}'-\mathbf{a}]=0$

or $\mathbf{b} \cdot \{\mathbf{d}\times(\mathbf{c}-\mathbf{a})\}=0$ and $\mathbf{b} \cdot \{\mathbf{d}'\times(\mathbf{c}'-\mathbf{a})\}=0$.

Above relation shows that $\mathbf{b}$ is perpendicular to both $\mathbf{d}\times(\mathbf{c}-\mathbf{a})$ and $[\mathbf{d}'\times(\mathbf{c}'-\mathbf{a})]$.

$\therefore$ $\mathbf{b}$ is parallel to $[\mathbf{d}\times(\mathbf{c}-\mathbf{a})]\times[\mathbf{d}'\times(\mathbf{c}'-\mathbf{a})]$

Hence required line is

$$\mathbf{r}=\mathbf{a}+k\,[\{\mathbf{d}\times(\mathbf{c}-\mathbf{a})\}\times\{\mathbf{d}'\times(\mathbf{c}'-\mathbf{a})\}].$$

(b) *Find the straight line, through the point* $\mathbf{c}$, *which is parallel to the plane* $\mathbf{r} \cdot \mathbf{a}=0$, *and intersects the line* $\mathbf{r}-\mathbf{a}'=t\mathbf{b}$.

(Agra 58)

Let the line be $\mathbf{r}=\mathbf{c}+t\mathbf{d}$ passing through $\mathbf{c}$.

It is parallel to $\mathbf{r} \cdot \mathbf{a}=0$ *i. e.* $\perp$ to normal $\mathbf{a}$.

$$\therefore\quad \mathbf{d} \cdot \mathbf{a}=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1)$$

It intersects $r=\mathbf{a}'+t\mathbf{b}$.

$$\therefore\quad [\mathbf{d}\ \mathbf{b}\ \mathbf{a}'-\mathbf{c}]=0 \quad \text{or} \quad \mathbf{d} \cdot [\mathbf{b}\times(\mathbf{a}'-\mathbf{c})]=0 \quad \ldots(2)$$

(1) and (2) show that $\mathbf{d}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}\times(\mathbf{a}'-\mathbf{c})$ and hence parallel to $\mathbf{a}\times\{(\mathbf{b}\times(\mathbf{a}'-\mathbf{c})\}$.

$\therefore$ line is $\mathbf{r}=\mathbf{c}+k\,[\mathbf{a}\times\{\mathbf{b}\times(\mathbf{a}'-\mathbf{c})\}]$.

Ex. 15. *Find the point of intersection of the planes*

$$\mathbf{r} \cdot n=q_1,\ \mathbf{r} \cdot \mathbf{n}_2=q_2,\ \mathbf{r} \cdot \mathbf{n}_3=q_3.$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ *are three given non-coplanar vectors i.e.* $[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]\neq 0$.

Since $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are three given non-coplanar vectors, $\mathbf{n}_1\times\mathbf{n}_2$, $\mathbf{n}_2\times\mathbf{n}_3$, $\mathbf{n}_3\times\mathbf{n}_1$ are also non-coplanar **Ex. 5. (2) P. 151** and we know that any vector can be expressed as a linear combination of any three non-coplanar vectors. Let the position vector $\mathbf{p}$ of the point of intersection be expressed as

$$\mathbf{p}=l\,(\mathbf{n}_2\times\mathbf{n}_3)+m\,(\mathbf{n}_3\times\mathbf{n}_1)+n\,(\mathbf{n}_1\times\mathbf{n}_2)$$

where l, m, n are to be found. Now $\mathbf{p}$ satisfies the equations of the three given planes and also noting that scalar triple product when two vectors are equal is zero.

$$\therefore\quad l\,(\mathbf{n}_2\times\mathbf{n}_3) \cdot \mathbf{n}_1=q \quad \text{or} \quad l=\frac{q_1}{[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]}.$$

Similarly $m=\frac{q_2}{[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]}$ and $n=\frac{q_3}{[\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3]}$. Hence $\mathbf{p}$ is etc.

Ex. 16. *A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Through A, B and C planes are drawn parallel to the coordinate planes. Prove that the locus of their point of intersection is given by*

$$x^{-2}+y^{-2}+z^{-2}=p^{-2}.$$

[See Author's Solid Geometry, Q. 7 (b) P. 61]

Let the equation to the plane be

$$\mathbf{r}\cdot\mathbf{n}=q \text{ where } p=\frac{q}{n} \qquad \ldots \quad \ldots(1)$$

and let $\mathbf{n}$ be $\qquad n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$.

$$\therefore \quad n^2=n_1^2+n_2^2+n_3^2 \qquad \ldots \quad \ldots(2)$$

Its intercept on the axis of x is

$$\frac{q}{\mathbf{i}\cdot\mathbf{n}}=\frac{q}{\mathbf{i}\cdot(n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k})}=\frac{q}{n_1}.$$

$$\therefore \quad A=\frac{q}{n_1}\mathbf{i}; \text{ similarly } B=\frac{q}{n_2}\mathbf{j} \text{ and } C=\frac{q}{n_3}\mathbf{k}.$$

Now any plane through A, *i.e.* $\frac{q}{n_1}\mathbf{i}$ and parallel to $\mathbf{j}$-$\mathbf{k}$ plane whose normal will be along $\mathbf{i}$ is given by

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0 \quad \text{or} \quad \mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$$

or
$$\mathbf{r}\cdot\mathbf{i}=\frac{q}{n_1}\mathbf{i}\cdot\mathbf{i} \quad \text{or} \quad \mathbf{r}\cdot\mathbf{i}=\frac{q}{n_1}.$$

Similarly planes through B, parallel to $\mathbf{k}$-$\mathbf{i}$-plane and plane through C parallel to $\mathbf{i}$-$\mathbf{j}$-plane are

$$\mathbf{r}\cdot\mathbf{j}=\frac{q}{n_2},\ \mathbf{r}\cdot\mathbf{k}=\frac{q}{n_3}.$$

Now if x, y, z be the coordinates of the point of intersection then $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ will satisfy the above three planes.

$$\therefore (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{i}=\frac{q}{n_1}, \quad \therefore \quad x=\frac{q}{n_1}.$$

Similarly $y=\dfrac{q}{n_2}$, $z=\dfrac{q}{n_3}$.

$\therefore \quad x^{-2}+y^{-2}+z^{-2}=\dfrac{n_1^2+n_2^2+n_3^2}{q^2}=\dfrac{n^2}{q^2}=p^{-2}$ from (1).

Ex. 17. *A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.*

[See Author's Solid Geometry, Q. 7 (a) P. 60]

If x, y, z be the centroid, then

$$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\frac{1}{4}\left[\frac{q}{n_1}\mathbf{i}+\frac{q}{n_2}\mathbf{j}+\frac{q}{n_3}\mathbf{k}+0\right].$$

Equating **i**, **j**, **k**, etc. we get x, y, and z etc.

Ex. 18. *Find the locus of a point which is equidistant from the three planes*

$\mathbf{r}\cdot\mathbf{n}_1=q_1$, $\mathbf{r}\cdot\mathbf{n}_2=q_2$, $\mathbf{r}\cdot\mathbf{n}_3=q_3$. **(Lucknow 51)**

Ans. $\dfrac{q_1-\mathbf{r}\cdot\mathbf{n}_1}{n_1}=\dfrac{q_2-\mathbf{r}\cdot\mathbf{n}_2}{n_2}=\dfrac{q_3-\mathbf{r}\cdot\mathbf{n}_3}{n_3}$.

Ex. 19. *OA, OB and OC are three mutually perpendicular lines; p is the length of the perpendicular from O to the plane ABC; show that $p^{-2}=a^{-2}+b^{-2}+c^{-2}$ and the area of the triangle ABC is $\frac{1}{2}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}$, a, b, c being the lengths of OA, OB and OC.*

Equation of the plane through **a**, **b**, **c** is

$$\mathbf{r}\cdot\mathbf{n}=[\mathbf{abc}] \text{ where } \mathbf{n}=\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$$

and its module being twice the area of the triangle whose vertices are **a**, **b** and **c** and perpendicular from O is

$$\frac{[\mathbf{abc}]}{|\mathbf{n}|}.$$

Here $\mathbf{a}=a\mathbf{i}$, $\mathbf{b}=b\mathbf{j}$ and $\mathbf{c}=c\mathbf{k}$ etc.

Ex. 20. *Prove that the four points $4\mathbf{i}+5\mathbf{j}+\mathbf{k}$, $-\mathbf{j}-\mathbf{k}$, $3\mathbf{i}+9\mathbf{j}+4\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}+\mathbf{k}$ are coplanar.*

Find $\mathbf{a}-\mathbf{d}$, $\mathbf{b}-\mathbf{d}$, $\mathbf{c}-\mathbf{d}$ and prove that their scalar triple product is zero.

Ex. 21. *Prove that the perpendicular distance of a point* **d** *from the plane through the points* **a**, **b** *and* **c** *is*

$$[\mathbf{abc}]-[\mathbf{abd}]+[\mathbf{acd}]-[\mathbf{bcd}]\div(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}).$$

Volume of tetrahedron whose vertices are **a, b, c, d** is $\frac{1}{6}[(\mathbf{abc})-(\mathbf{abd})+(\mathbf{acd})-(\mathbf{bcd})]=\frac{1}{3}$ area of Δ through **a, b, c** $\times$ perpendicular distance of the point *d* from the plane through **a, b, c**$=\frac{1}{3}\cdot\frac{1}{2}(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})\times p$.

$\therefore$ p=as given etc.

Ex. 22. *Prove that the shortest distance between opposite edges of a regular tetrahedron is equal to half the diagonal of the square described on an edge.* **(Agra 50, 59)**

We know that a regular tetrahedron can be inscribed in a cube.

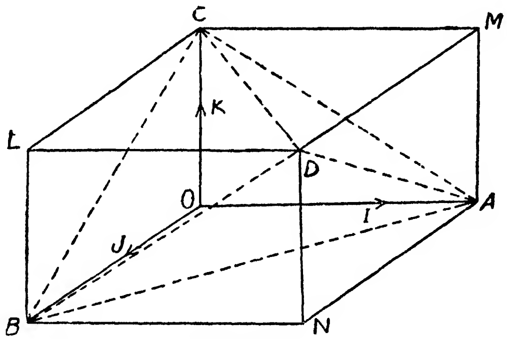


Fig No 97

Let the unit vectors along *OA*, *OB*, *OC* be **i**, **j** and **k** respectively. $\therefore$ $\overrightarrow{OD}=\mathbf{i}+\mathbf{j}+\mathbf{k}$. *BACD* is the regular tetrahedron and we are to find the shortest distance between pair of opposite edges say *BC* and *AD*.

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\mathbf{k}-\mathbf{j}$$

and $$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(\mathbf{i}+\mathbf{j}+\mathbf{k})-\mathbf{i}=\mathbf{j}+\mathbf{k}.$$

$\therefore \quad \overrightarrow{BC} \times \overrightarrow{AD} = (\mathbf{k}-\mathbf{j}) \times (\mathbf{k}+\mathbf{j}) = \mathbf{k} \times \mathbf{j} - \mathbf{j} \times \mathbf{k} = -2\mathbf{i}.$

Module of $\overrightarrow{BC} \times \overrightarrow{AD} = |-2\mathbf{i}| = 2.$

Now B is any point on BC and A a point on AD.

$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{j} - \mathbf{i}.$

Required shortest distance is the projection of AB on

$\overrightarrow{BC} \times \overrightarrow{AD}$. $\therefore$ S.D. $= \dfrac{\overrightarrow{AB} \cdot (\overrightarrow{BC} \times \overrightarrow{AD})}{|\overrightarrow{BC} \times \overrightarrow{AD}|}$ [§ 11 P. 178]

$$= \frac{(\mathbf{j}-\mathbf{i}) \cdot -2\mathbf{i}}{2} = 1,$$

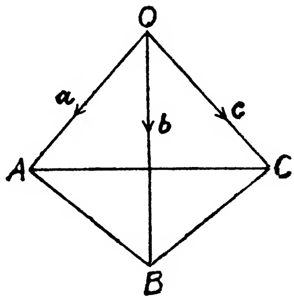
Now $\qquad AB = |\mathbf{j}-\mathbf{i}| = \sqrt{2}.$

$\therefore$ diagonal of the square with one side AB

$$= \sqrt{(2+2)} = 2.$$

$\therefore$ S.D. $= 1 = \frac{2}{2} = \frac{1}{2}$. the diagonal of square.

Alternative Method.



Let $OABC$ be the regular tetrahedron and O be the origin of vectors with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ the position vectors of A, B, and C. The two opposite edgs are OA and BC.

The equations of these are

$\mathbf{r} = t\mathbf{a}$ and $\mathbf{r} = (1-s)\,\mathbf{b} + s\mathbf{c}$

$$= \mathbf{b} + s\,(\mathbf{c}-\mathbf{b}).$$

The S.D. between them is

$$p = \frac{\mathbf{a} \times (\mathbf{c}-\mathbf{b}) \cdot \mathbf{b}}{|\mathbf{a} \times (\mathbf{c}-\mathbf{b})|}$$

$$= \frac{\mathbf{a} \times (\mathbf{c}-\mathbf{b}) \cdot \mathbf{b}}{|\mathbf{a} \times \mathbf{c} - \mathbf{a} \times \mathbf{b}|} \qquad \ldots \quad \ldots \quad \ldots (1)$$

$\mathbf{a}\times\mathbf{c}=2\Delta_1\, \overset{\wedge}{OAC}\,.\,\mathbf{n}_1$ where $\overset{\wedge}{\mathbf{n}_1}$ is the unit vector perpendicular to the $\triangle OAC$.

$\mathbf{a}\times\mathbf{b}=2\Delta_2\, \overset{\wedge}{OAB}\,.\,\mathbf{n}_2$ where $\overset{\wedge}{\mathbf{n}_2}$ is the unit vector perpendicular to the $\triangle OAB$.

Module of $\mathbf{a}\times\mathbf{c}-\mathbf{a}\times\mathbf{b}$=module of $2\Delta_1\overset{\wedge}{\mathbf{n}_1}-2\Delta_2\overset{\wedge}{\mathbf{n}_2}$

$$=\sqrt{[4\Delta^2\,(\mathbf{n}_1{}^2+\mathbf{n}_2{}^2-2\mathbf{n}_1\,.\,\mathbf{n}_2)]},$$

as $\Delta_1=\Delta_2$ being areas of the faces of the regular tetrahedron

$$=\sqrt{[4\Delta^2\,(1+1-2\cos\theta)}$$

where θ is the angle between the two faces and hence $\cos\theta=\frac{1}{3}$.

$\therefore$ module of $\mathbf{a}\times\mathbf{c}-\mathbf{a}\times\mathbf{b}=\sqrt{(8\Delta^2\,.\,\frac{2}{3})}=\dfrac{4\Delta}{\sqrt{3}}$ ... ...(2)

Hence $p=\dfrac{[\mathbf{abc}]}{\dfrac{4\Delta}{\sqrt{3}}}=\dfrac{6V}{\dfrac{4}{3}\,.\,\dfrac{a^2\sqrt{3}}{4}}=\dfrac{6V}{a^2}$ [from (1)]

$$=6\,.\,\frac{a^3\sqrt{2}}{12}\,.\,\frac{1}{a^2}=\frac{a}{\sqrt{2}},\qquad \because\ V=\frac{a^3\sqrt{2}}{12}.$$

$$p=\frac{a}{\sqrt{2}}\,.\,\frac{1}{2}\,.\,a\sqrt{2}=\frac{1}{2}\sqrt{(a^2+a^2)}$$

i.e. half the diagonal of the square described on an edge.

Ex. 23. *Prove that the S. D. between pairs of opposite edges of an isosceles tetrahedron lie along the join of their mid. points and that the three S.D.'s are perpendicular.*

Take one vertex as origin O and the position vectors of the other be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

$\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$,

$$\overrightarrow{PQ}=\frac{\mathbf{b}+\mathbf{c}}{2}-\frac{\mathbf{a}}{2}$$

$$=\frac{\mathbf{b}+\mathbf{c}-\mathbf{a}}{2}.$$

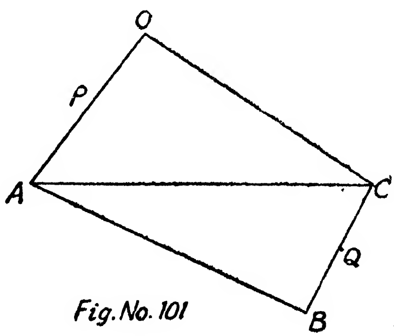


Fig. No. 101

If *PQ* is S. D. between *OA* and *BC* then *PQ* should be perpendicular to both *OA* and *BC* and it will be so if

$$\tfrac{1}{2}(\mathbf{b}+\mathbf{c}-\mathbf{a})\cdot\mathbf{a}=0 \text{ and } \tfrac{1}{2}(\mathbf{b}+\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})=0$$

or if $$\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=a^2 \quad \ldots \quad \ldots \quad \ldots(1)$$

and $$c^2-b^2=\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b} \quad \ldots \quad \ldots \quad \ldots(2)$$

Since the tetrahedron is isosceles, we have

$$OA=BC,\ AB=CO,\ CA=OB$$

or $$a^2=(\mathbf{c}-\mathbf{b})^2 \quad \text{or} \quad a^2=b^2+c^2-2\mathbf{b}\cdot\mathbf{c}$$

or $$2\mathbf{b}\cdot\mathbf{c}=(b^2+c^2-a^2) \quad \ldots \quad \ldots \quad \ldots(3)$$

Similarly $$2\mathbf{c}\cdot\mathbf{a}=(c^2+a^2-b^2) \quad \ldots \quad \ldots \quad \ldots(4)$$

and $$2\mathbf{a}\cdot\mathbf{b}=(a^2+b^2-c^2) \quad \ldots \quad \ldots \quad \ldots(5)$$

Adding (4) and (5), we get (1) and subtracting (4) and (5) we get (2).

Hence *PQ* is S. D. between *OA* and *BC*.

Similarly we can prove for other pairs of opposite edges. The three S. D.'s lie along

$$\mathbf{b}+\mathbf{c}-\mathbf{a},\ \mathbf{c}+\mathbf{a}-\mathbf{b},\ \mathbf{a}+\mathbf{b}-\mathbf{c}$$

and these will be perpendicular if

$$(\mathbf{b}+\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}+\mathbf{a}-\mathbf{b})=0 \text{ etc.}$$

if $$c^2-(\mathbf{a}-\mathbf{b})^2=0$$

or if $$2\mathbf{a}\cdot\mathbf{b}=a^2+b^2-c^2 \text{ which is true by (5).}$$

Similarly we can prove other pairs to be perpendicular by (3) and (4).

§ 12. General equation of a sphere.

(Agra 35, 38, 43, 52, 60)

If *C* be the centre of a sphere, then by definition, the distance of any point on the surface of the sphere from the centre is equal to radius *a*.

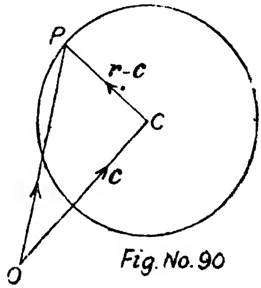


Fig. No. 90

Let the position vector of any point *P* on the surface be $\mathbf{r}$ with respect to an origin *O* and that of

the centre be **c**, *i.e.* $\overrightarrow{OP}=\mathbf{r}$ and $\overrightarrow{OC}=\mathbf{c}$; then

$$\overrightarrow{CP}=\overrightarrow{OP}-\overrightarrow{OC}=\mathbf{r}-\mathbf{c}.$$

Now module of $\overrightarrow{CP}$=radius=**a** and we know that square of a vector is square of its module.

$$\therefore \quad (\mathbf{r}-\mathbf{c})^2=a^2$$

or $$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+c^2-a^2=0 \quad \ldots \quad \ldots \quad \ldots(1)$$

$$\because \quad \mathbf{c}^2=c^2.$$

We may for the sake of convenience put $c^2-a^2=k$ and the above equation becomes

$$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k=0 \quad \ldots \quad \ldots \quad \ldots(2)$$

Above is a relation between the position vector of any point on the surface of the sphere and is called the equation of the sphere.

Corresponding cartesian form.

If P be the point (x, y, z) and C $(c_1, c_2, {}_3)$, then resolving in terms of unit vectors **i**, **j** and **k**,

$$\overrightarrow{OP}=\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

and $$\overrightarrow{OC}=\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

$$\therefore \quad \overrightarrow{CP}=\mathbf{r}-\mathbf{c}=(x-c_1)\,\mathbf{i}+(y-c_2)\,\mathbf{j}+(z-c_3)\,\mathbf{k}.$$

Squaring both sides, we get

$$a^2=(x-c_1)^2+(y-c_2)^2+(z-c_3)^2 \quad \because \quad |\overrightarrow{CP}|^2=a^2$$

which is the well known cartesian equation of the sphere whose centre is at (c_1, c_2, c_3) and radius is a.

Particular case.

1. In case the origin of vectors lies on the sphere.

In this case the module of $\overrightarrow{OC}$ will be equal to radius *i. e.* $c=a$ and hence from (1), we get the equation of the sphere as

$$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots(3)$$

$$\because \quad k=c^2-a^2=a^2=a^2=0.$$

Polar form. $\mathbf{r}^2=r^2$ and $\mathbf{r}\cdot\mathbf{c}=rc\cos\theta=ra\cos\theta$

$\therefore$ (3) gives $r^2-2ra\cos\theta=0$ or $r=2a\cos\theta$ is the required polar form.

Cartesian form.

Putting the values of $\mathbf{r}$ and $\mathbf{c}$ in terms of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we get from (3),

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})^2=2\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})$$

or

$$x^2+y^2+z^2=2\,(c_1x+c_2y+c_3z)$$

which represents a sphere whose centre is at (c_1, c_2, c_3) and which passes through origin.

2. If the centre is at the origin.

In this case O is at the point C and hence clearly the equation of the sphere takes the form $\mathbf{r}^2=\mathbf{a}^2$.

or

$$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}+\mathbf{a})=0 \ldots \quad \ldots \quad \ldots \quad \ldots(4)$$

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\mathbf{r}-\mathbf{a},$$

$$\overrightarrow{BP}=\overrightarrow{OP}-\overrightarrow{OB}=\mathbf{r}+\mathbf{a}.$$

$\because$ positon vector of B is $-\mathbf{a}$,

$$\overrightarrow{AP}\cdot\overrightarrow{BP}=(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}+\mathbf{a})=0$$

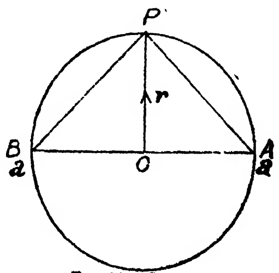


Fig. No. 91

and we know that if dot product of two vectors be zero, then they are perpendicular. Thus AP is perpendicular to BP, *showing that diameter of a sphere subtends a right angle at the surface.*

General Method for above property. **(Agra 52)**

Let the equation of the sphere be

$$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k=0 \text{ where } k=c^2-a^2 \qquad \ldots \quad \ldots(5)$$

If $\vec{CA}=\mathbf{a}$, then $\vec{CB}=-\mathbf{a}$.

$\therefore \quad \vec{OA}=\vec{OC}+\vec{CA}=\mathbf{c}+\mathbf{a}$,

$\vec{OB}=\vec{OC}+\vec{CB}=\mathbf{c}-\mathbf{a}$.

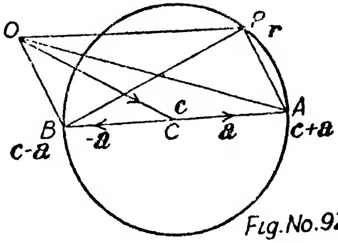


Fig. No. 92

Above gives us the position vectors of A and B.

If $\mathbf{r}$ be the position vector of any point P on the circumference, then

$$\vec{AP}=\vec{OP}-\vec{OA}=\mathbf{r}-(\mathbf{c}+\mathbf{a})$$

and $$\vec{BP}=\vec{OP}-\vec{OB}=\mathbf{r}-(\mathbf{c}-\mathbf{a}).$$

Now $\quad \vec{AP}\cdot\vec{BP}=(\mathbf{r}-\mathbf{c}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{c}+\mathbf{a})$

or $\quad (\mathbf{r}-\mathbf{c})^2-\mathbf{a}^2$ or $\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+c^2-a^2$

or $\quad \mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k$ which is zero by (5).

Hence AP is perpendicular to BP, **showing that a diameter subtends a right angle at the circumference.**

Cor. 1. Equation of a sphere on the join of two given points as diameter. **(Benaras 48, 54)**

If $\mathbf{g}$ and $\mathbf{h}$ be the position vectors of the extremities G and H of any diameter and P any point on the surface whose position vector is $\mathbf{r}$, then $\angle GPH=\pi/2$, *i. e.* GP is perpendicular to HP.

$\therefore$ dot product of $\vec{GP}$ and $\vec{GH}$ is zero

i. e. $$(\mathbf{r}-\mathbf{g})\cdot(\mathbf{r}-\mathbf{h})=0.$$

Above represents the required equation of the sphere.

§ 13. Points of intersection of a line and a sphere.

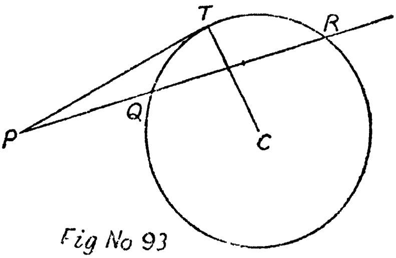


Fig No 93

Let the equation of the sphere be

$$F(\mathbf{r})=\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k=0 \qquad \ldots.(1)$$

where $k=c^2-a^2$.

Let the line pass through the point P whose position vector is $\mathbf{p}$ and be a parallel to unit vector $\hat{\mathbf{q}}$, so that its equation is

$$\mathbf{r}=\mathbf{p}+t\hat{\mathbf{q}} \ldots \quad \ldots(2) \qquad [\text{§ 6, P. 46}]$$

t will stand for the distance of the point P from any point on the line.

In order to find the points of intersection we have to eliminate $\mathbf{r}$ between (1) and (2) and noting that square of a unit vector is unity, we get, on putting the value of $\mathbf{r}$ from (2) in (1),

$$(\mathbf{p}+t\,\hat{\mathbf{q}})^2-2(\mathbf{p}+t\,\hat{\mathbf{q}})\cdot\mathbf{c}+k=0$$

or $$t^2+2t\,\hat{\mathbf{q}}\cdot\mathbf{p}+\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{c}-2t\,\hat{\mathbf{q}}\cdot\mathbf{c}+k=0$$

or $$t^2+2\hat{\mathbf{q}}\cdot(\mathbf{p}-\mathbf{c})\,t+(\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{c}+k)=0$$

or $$t^2+2\hat{\mathbf{q}}\cdot(\mathbf{p}-\mathbf{c})\,t+F(\mathbf{p})=0 \ldots \quad \ldots \quad \ldots(3)$$

Above being a quadratic in t shows that every line cuts the surface of sphere in two points which will be real if

$$[\hat{\mathbf{q}}\cdot(\mathbf{p}-\mathbf{c})]^2 \geqslant F(\mathbf{p}), \; i.\,e.\; B^2 \geqslant 4AC.$$

If Q and R are the points of intersection, then the roots of the above equation (3) will be the values of distances of P from Q and R, *i.e.* PQ and PR.

$\therefore\quad PQ \cdot PR =$ Product of roots $= F\,(\mathbf{p})$.

Above result dces not depend upon the direction of lines as it is independent of $\hat{\mathbf{q}}$ showing thereby $PQ \cdot PR = F\,(\mathbf{p})$ for all lines drawn in any direction through the point P to cut the surface of the sphere.

Cor. 2. Square of tangent from any point.

In case Q and R coincide at any point T, then PQR becomes tangent line PT and both PQ and PR become PT.

$$\therefore\quad PT \cdot PT = F\,(\mathbf{p}) \quad \text{or} \quad PT^2 = F\,(\mathbf{p}).$$

Thus square of the tangent from any point to the surface of a sphere is obtained by substituting that point in the equation of the sphere, which is a result that is identical with the corresponding result of coordinate geometry.

Cor. 3. Tangent plane at a given point. [*Refer Author's Coordinate Solid Geometry.*] *The same procedure is adopted for finding tangent planes at any point to a conicoid.*

Now let us choose that the point P is on the surface of the sphere so that $F\,(\mathbf{p})=0$ and then one root of (3) will be zero as PQ will be zero in this case. If the line through P is to be a tangent line then the other root of (3) should also be zero, the condition for which is

$$\hat{\mathbf{q}} \cdot (\mathbf{p}-\mathbf{c})=0 \text{ from (3)} \qquad \ldots \quad \ldots(4)$$

Now $\hat{\mathbf{q}}$ being the direction of line which is now a tangent line and $\mathbf{p}-\mathbf{c}$ is the vector joining the centre to the point P and since their dot product is zero we conclude **that tangent line is perpendicular to the radius through that point.**

All such tangent lines will lie in a plane whose equation is obtained by eliminating $\hat{\mathbf{q}}$ between (2) and (4) (in solid geometry by eliminating $l, m, n,$ *i.e.* direction cosines) and we get the equation of the tangent plane as

$$(\mathbf{r}-\mathbf{p}) \cdot (\mathbf{p}-\mathbf{c})=0. \qquad \ldots \quad \ldots(5)$$

Above equation clearly represents a plane through the point $\mathbf{p}$ whose normal is $\mathbf{p}-\mathbf{c}$ *i.e.* the line joining the centre and the point of contact.

Now we know that $F(\mathbf{p})=0$ and hence the equation (5) remains unchanged if we add $F(\mathbf{p})$ in L. H. S. Thus the tangent plane becomes

$$(\mathbf{r}-\mathbf{p}) \cdot (\mathbf{p}-\mathbf{c})+F(\mathbf{p})=0$$

or
$$\mathbf{r} \cdot \mathbf{p}-\mathbf{p}^2-\mathbf{r} \cdot \mathbf{c}+\mathbf{p} \cdot \mathbf{c}+\mathbf{p}^2-2\mathbf{p} \cdot \mathbf{c}+k=0$$

or
$$\mathbf{r} \cdot \mathbf{p}-(\mathbf{r}+\mathbf{p}) \cdot \mathbf{c}+k=0 \qquad \ldots \quad \ldots(6)$$

Above is the equation of the tangent plane at $\mathbf{p}$.

Rule. *In the equation of the sphere replace* $\mathbf{r}^2$ *by* $\mathbf{r} \cdot \mathbf{r}$ *and change one of the* $\mathbf{r}$*'s by the given point* $\mathbf{p}$ *and replace* $2\mathbf{r}$ *by* $\mathbf{r}+\mathbf{r}$ *and change one of the* $\mathbf{r}$*'s by the given point* $\mathbf{p}$ *and this rule is identical with the corresponding rule of coordinate geometry.*

Cor. 4. Condition for any plane to be a tangent plane.

We have seen that tangent plane at any point is perpendicular to the radius through that point and as such if any plane is a tangent plane then its perpendicular distance from the centre should be equal to radius.

Let the plane be $\mathbf{r} \cdot \mathbf{n}=q$ and $\mathbf{c}$ be the centre and a the radius.

$$\therefore \left(\frac{q-\mathbf{c} \cdot \mathbf{n}}{n}\right)^2=a^2. \qquad [\S\ 5\ \text{P. } 166]$$

Ex. 1. *Find the coordinates of the centre of the sphere inscribed in the tetrahedron bounded by the planes*

$$\mathbf{r}\cdot\mathbf{i}=0,\ \mathbf{r}\cdot\mathbf{j}=0,\ \mathbf{r}\cdot\mathbf{k}=0 \text{ and } \mathbf{r}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=a.$$

Also write down the equation of the sphere. **(Benaras 53)**

Let (x, y, z) be the coordinates of the centre so that the position vector of the point is $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Since the given planes are all tangent planes, therefore, perpendiculars from the centre to all the planes are equal.

$$\frac{0-(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{i}}{|\mathbf{i}|}=\frac{0-(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{j}}{|\mathbf{j}|}$$

$$=\frac{0-(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{k}}{|\mathbf{k}|}=\frac{a-(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|}.$$

Since perpendicular distance is not $-$ive, we get

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=\frac{a-(x+y+z)}{\sqrt{3}}=k \text{ (say)}.$$

$$\therefore\quad x=k,\ y=k,\ z=k \text{ and } a-(k+k+k)=\sqrt{3}k$$

or

$$a=k(\sqrt{3}+3)$$

or

$$k=\frac{a}{3+\sqrt{3}}=\frac{a(3-\sqrt{3})}{9-3}=\frac{a}{6}(3-\sqrt{3}).$$

$$\therefore\quad x=y=z=\frac{a}{6}(3-\sqrt{3})=\text{radius}.$$

Hence on putting the values of x, y and z the position vector of the centre is

$$\frac{a}{6}(3-\sqrt{3})(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

Hence the equation of the sphere is $(\mathbf{r}-\mathbf{c})^2=\mathbf{a}^2$.

$$\left[\mathbf{r}-\frac{a}{6}(3-\sqrt{3})(\mathbf{i}+\mathbf{j}+\mathbf{k})\right]^2=\left[\frac{a}{6}(3-\sqrt{3})\right]^2.$$

(b) *Prove that the equation of the sphere circumscribing the tetrahedron of part (a) is* $\mathbf{r}\cdot\{\mathbf{r}-a(\mathbf{i}+\mathbf{j}+\mathbf{k})\}=0.$

Cor. 5. Condition for orthogonal intersection of two spheres.

In case the two spheres cut each other orthogonally, then evidently the tangent plane to one of them at their common point of intersection will pass through the centre of the other. Hence the square of the distance between their centres should be equal to sum of the squares of their radii, *i.e.* if the two spheres be

$$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}_1+k_1=0, \text{ where } k_1=c_1^2-a^2$$

and $$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}_2+k_2=0, \text{ where } k_2=c_2^2-a^2,$$

then $$(\mathbf{c}_1-\mathbf{c}_2)^2=a_1^2+a_2^2$$

or $$c_1^2+c_2^2-2\mathbf{c}_1\cdot\mathbf{c}_2=c_1^2-k_1+c_2^2-k_2$$

or $$2\mathbf{c}_1\cdot\mathbf{c}_2=k_1+k_2.$$

Corresponding cartesian result.

Two spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

and $$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

will cut one another orthogonally if

$$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2.$$

[Refer Author's Solid Geometry P. 213]

Ex. 2. *The sphere which cuts $F_1(\mathbf{r})=0$ and $F_2(\mathbf{r})=0$ orthogonally also cuts $F_1(\mathbf{r})-\lambda F_2(\mathbf{r})=0$ orthogonally.*

(Alld. M. Sc. 1960, Agra 38, 46; Benaras 55)

Let $$F_1(\mathbf{r})=\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}_1+k_1=0 \qquad \ldots \quad \ldots(1)$$

$$F_2(\mathbf{r})=\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}_2+k_2=0 \qquad \ldots \quad \ldots(2)$$

$$\therefore \quad F_1(r)-\lambda F_2(r)=(\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}_1+k_1)-\lambda(\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}_2+k_2)=0$$
$$=\mathbf{r}^2(1-\lambda)-2\mathbf{r}\cdot(\mathbf{c}_1-\lambda\mathbf{c}_2)+k_1-\lambda k_2=0$$

or $$\mathbf{r}^2-2r\cdot\left(\frac{\mathbf{c}_1-\lambda\mathbf{c}_2}{1-\lambda}\right)+\frac{k_1-\lambda k_2}{1-\lambda}=0 \qquad \ldots \quad \ldots(3)$$

Let the sphere $\mathbf{r}^2 - 2\mathbf{r} \cdot \mathbf{c} + k = 0$ ... ...(4)
cut the spheres (1) and (2) orthogonally.

$$2\mathbf{c} \cdot \mathbf{c}_1 = k + k_1. \qquad \ldots \quad \ldots(5)$$

$$\therefore \quad 2\mathbf{c} \cdot \mathbf{c}_2 = k + k_2. \qquad \ldots \quad \ldots(6)$$

Multiplying (6) by λ and subtracting from (5), we get

$$2\mathbf{c} \cdot (\mathbf{c}_1 - \lambda \mathbf{c}_2) = (k_1 - \lambda k_2) + k\,(1 - \lambda)$$

or

$$2\mathbf{c} \cdot \left(\frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda}\right) = \frac{k_1 - \lambda k_2}{1 - \lambda} + k.$$

Above is evidently the condition that the sphere (4) may cut sphere (3) orthogonally.

§ 14. The polar plane.

Defi. The polar plane of a given point with respect to a sphere is the locus of points the tangent planes at which pass through the given point.

Let the equation of the sphere be

$$F(\mathbf{r}) = \mathbf{r}^2 - 2\mathbf{r} \cdot \mathbf{c} + k = 0.$$

The tangent plane at any point $\mathbf{p}$ is given by

$$\mathbf{r} \cdot \mathbf{p} - (\mathbf{r} + \mathbf{p}) \cdot \mathbf{c} + k = 0.$$

If it passes through a given point $\mathbf{d}$ say, then

$$\mathbf{d} \cdot \mathbf{p} - (\mathbf{d} + \mathbf{p}) \cdot \mathbf{c} + k = 0.$$

The locus of point P is therefore

$$\mathbf{r} \cdot \mathbf{d} - (\mathbf{d} + \mathbf{r}) \cdot \mathbf{c} + k = 0 \qquad \ldots \quad \ldots(1)$$

or it can be written as

$$\mathbf{r} \cdot (\mathbf{d} - \mathbf{c}) = (\mathbf{c} \cdot \mathbf{d} - k) \qquad \ldots \quad \ldots \quad \ldots(2)$$

Above equation represents a plane which is clearly perpendicular to the line joining the centre and the given point.

Thus the polar plane of a point is perpendicular to the line joining the centre of the sphere to that point.

Again let the polar plane of point $\mathbf{d}$ cut the line joining the centre to $\mathbf{d}$ in T : then CT is the perpendicular distance

of $\mathbf{c}$ from the polar plane of $\mathbf{d}$ as this line is normal to the polar plane.

$$\therefore \quad CT = \frac{\{(\mathbf{c}\,.\,\mathbf{d}) - k\} - \mathbf{c}\,.\,(\mathbf{d}-\mathbf{c})}{|\,\mathbf{d}-\mathbf{c}\,|} \; [\S\ 5\ \mathbf{P.}\ 166]\ i.e.\ \frac{q - \mathbf{r}'\,.\,\mathbf{n}}{n}$$

$$= \frac{-k + c^2}{CD} = \frac{a^2}{CD}$$

where D is the point whose position vector is $\mathbf{d}$.

$$\therefore \quad CD\,.\,CT = a^2.$$

The two points D and T are called inverse points with respect to the sphere.

Also it is easy to prove from (1) that if the polar plane of a point $\mathbf{d}$ passes through the point $\mathbf{e}$, then the polar plane of $\mathbf{e}$ will pass through the point $\mathbf{d}$.

§ 15. **Radical plane.**

The radical plane of any two given spheres is the plane which contains all such points the squares of the tangents from which to the given spheres are equal.

Let the two spheres be

$$\mathbf{r}^2 - 2\mathbf{r}\,.\,\mathbf{c}_1 + k_1 = 0 \qquad \ldots \quad \ldots \quad \ldots (1)$$

$$\mathbf{r}^2 - 2\mathbf{r}\,.\,\mathbf{c}_2 + k_2 = 0 \qquad \ldots \quad \ldots \quad \ldots (2)$$

Let there be a point $\mathbf{p}$ such that squares of the tangents from it to (1) and (2) are equal.

$$\therefore \quad \mathbf{p}^2 - 2\mathbf{p}\,.\,\mathbf{c}_1 + k_1 = \mathbf{p}^2 - 2\mathbf{p}\,.\,\mathbf{c}_2 + k_2 = 0$$

or
$$2\mathbf{p}\,.\,(\mathbf{c}_1 - \mathbf{c}_2) = k_1 - k_2.$$

Above shows that the point $\mathbf{p}$ lies on the plane

$$2\mathbf{r}\,.\,(\mathbf{c}_1 - \mathbf{c}_2) = k_1 - k_2 \qquad \ldots \quad \ldots \quad \ldots (3)$$

Clearly the above plane is perpendicular to $\mathbf{c}_1 - \mathbf{c}_2$, *i.e.* the line joining the centres.

Rule. The radical plane of two given spheres is obtained by subtracting the equations of the spheres as can be seen from (1), (2) and (3).

Ex. 3. *The locus of a point which moves so that its distances from two fixed points are in a constant ratio $n : 1$ is a sphere. Prove also that all such spheres, for different values of n, have a common radical plane.*

Let the middle point of the line joining the fixed points A and B be origin so that position vector of A is $\mathbf{a}$ and of B is $-\mathbf{a}$. Let the point P be $\mathbf{r}$.

$$\therefore \quad AP=\mathbf{r}-\mathbf{a} \text{ and } \overrightarrow{BP}=\mathbf{r}+\mathbf{a}.$$

$$\therefore \quad AP^2=(\mathbf{r}-\mathbf{a})^2 \text{ and } BP^2=(\mathbf{r}+\mathbf{a})^2.$$

Also we are given that

$$\frac{AP}{BP}=\frac{n}{1} \quad \text{or} \quad AP^2=n^2 \,.\, BP^2.$$

$$\therefore \quad (\mathbf{r}^2-2\mathbf{r}\,.\,\mathbf{a}+a^2)=n^2\,(\mathbf{r}^2+2\mathbf{r}\,.\,\mathbf{a}+a^2)$$

or,

$$\mathbf{r}^2\,(1-n^2)-2\mathbf{r}\,.\,\mathbf{a}\,.\,(1+n^2)+a^2\,(1-n^2)=0$$

or

$$\mathbf{r}^2-2\mathbf{r}\,.\,\mathbf{a}\left(\frac{1+n^2}{1-n^2}\right)+a^2=0.$$

Above equation clearly represents a sphere.

Giving n the values n_1 and n_2, we get two spheres.

$$\mathbf{r}^2-2\mathbf{r}\,.\,\mathbf{a}\,\frac{1+n_1^2}{1-n_1^2}+a^2=0,$$

$$\mathbf{r}^2-2\mathbf{r}\,.\,\mathbf{a}\,\frac{1+n_2^2}{1-n_2^2}+a^2=0.$$

The radical plane of the above sphere is obtained by subtracting them and is

$$2\mathbf{r}\,.\,\mathbf{a}\left\{\frac{1+n_1^2}{1-n_1^2}-\frac{1+n_2^2}{1-n_2^2}\right\}=0 \quad \text{or} \quad \mathbf{r}\,.\,\mathbf{a}=0$$

which is independent of n_1 and n_2.

Above represents the radical plane which passes through origin which is the middle point of AB and the normal is along $\mathbf{a}$.

Thus the radical plane bisects perpendicularly the distance between the given points.

Exercises

Ex. 1. *Prove that the locus of a point the sum of the squares of whose distances from a given point is constant is a sphere whose centre is at the centroid of the given points.*

Ex. 2. *Prove that the distances of two points from the centre of a given sphere are proportional to the distances of the points each from the polar plane of the other.*

For the sake of convenience let us choose the equation of the sphere as $\mathbf{r}^2=\mathbf{a}^2$, centre being at origin and the given points as $\mathbf{a}_1$ and $\mathbf{a}_2$, whose polar planes are

$$\mathbf{r}\cdot\mathbf{a}_1=a^2 \text{ and } \mathbf{r}\cdot\mathbf{a}_2=a^2 \text{ etc.}$$

Ex. 3. *From any point on the surface of a sphere, straight lines are drawn to extremities of any diameter of a concentric sphere. Prove that sum of the squares on these lines is constant.*

(Agra 38)

Let the centre of the concentric spheres be the point $\mathbf{c}$ so that the equations of the outer and inner spheres are

$$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k_1=0 \ldots \quad \ldots(1)$$

$$\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k_2=0 \ldots \quad \ldots(2)$$

where $k_1=c^2-a_1^2$ and $k_2=c^2-a_2^2 \ldots(2)$

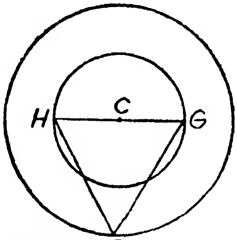


Fig. No. 99

Let $\mathbf{r}_1$ be any point on the outer sphere so that

$$\mathbf{r}_1^2-2\mathbf{r}_1\cdot\mathbf{c}+k_1=0 \quad \ldots \quad \ldots \quad \ldots(3)$$

If $\mathbf{g}$ and $\mathbf{h}$ be extremities of diameter of the inner sphere, then its equation is $(\mathbf{r}-\mathbf{g})\cdot(\mathbf{r}-\mathbf{h})=0$

or $$\mathbf{r}^2-\mathbf{r}\cdot(\mathbf{g}+\mathbf{h})+\mathbf{g}\cdot\mathbf{h}=0 \quad \ldots \quad \ldots \quad \ldots(4)$$

Comparing (2) and (4), we get

$$\mathbf{g}+\mathbf{h}=2\mathbf{c} \text{ and } \mathbf{g}\cdot\mathbf{h}=k_2 \quad \ldots \quad \ldots \quad \ldots(5)$$

Now $PG^2+PH^2=\vec{PG}^2+\vec{PH}^2=(\mathbf{g}-\mathbf{r}_1)^2+(\mathbf{h}-\mathbf{r}_1)^2.$

$$=\mathbf{g}^2+\mathbf{h}^2-2\mathbf{r}_1\cdot(\mathbf{g}+\mathbf{h})+2\mathbf{r}_1^2$$
$$=(\mathbf{g}+\mathbf{h})^2-2\mathbf{g}\cdot\mathbf{h}-2\mathbf{r}_1\cdot(\mathbf{g}+\mathbf{h})+2\mathbf{r}_1^2$$
$$=4\mathbf{c}^2-2k_2-2\mathbf{r}_1\cdot 2\mathbf{c}+2\mathbf{r}_1^2$$

or $$4c^2-2k_2+2(\mathbf{r}_1^2-2\mathbf{r}_1\cdot\mathbf{c})$$
$$=4c^2-2k_2+2k_1 \qquad \text{[from (3)]}$$
$$=2[(c^2-k_1)+(c^2-k_2)]=2(a_1^2+a_2^2)$$
= twice the sum of the squares of the radii and hence constant.

Ex. 4. *A straight line is drawn from a point O to meet a fixed sphere in P. In OP a point Q is taken so that OP : OQ is a fixed ratio. Prove that the locus of Q is a sphere.* **(Agra 35)**

Let $\mathbf{r}_1$ be the position vector of Q on OP, so that P will be $n\mathbf{r}_1$, where n is constant and this point P lies on the sphere and hence the locus of Q *i.e.* $\mathbf{r}_1$ is etc. etc.

Ex. 5. *A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Prove by vectors that the locus of the centre of the sphere OABC is* $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$

[See Author's Solid Geometry Q. 4 (a) P. 165]

The fixed point $A=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and let the normal to the plane be $\mathbf{n}=n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$.

The equation of the plane through A is given by

$$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$$

or $$\mathbf{r}\cdot(n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k})=an_1+bn_2+cn_3=q \text{ say.}$$

If x_1 be the intercept on the axis of x, then $x_1\mathbf{i}$ lies on the plane.

$$\therefore\quad x_1\mathbf{i}\cdot(n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k})=q \quad\text{or}\quad x_1n_1=q \quad\text{or}\quad x_1=\frac{q}{n_1}.$$

$\therefore$ point A is $\frac{q}{n_1}\mathbf{i}$. Similarly B is $\frac{q}{n_2}\mathbf{j}$ and C is $\frac{q}{n_3}\mathbf{k}$.

The point O is origin.

Let the coordinates of the centre be $P(x, y, z)$, *i.e.*, $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

$\therefore \quad OP=AP=BP=CP=$radius of the sphere $OABC$

$$\left|x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right|=\left|\left(x-\frac{q}{n_1}\right)\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right|=\left|x\mathbf{i}+\left(y-\frac{q}{n_2}\right)\mathbf{j}+z\mathbf{k}\right|$$
$$=\left|x\mathbf{i}+y\mathbf{j}+\left(z-\frac{q}{n_3}\right)\mathbf{k}\right|.$$

Equating their modules, we get

$$x^2+y^2+z^2=\left(x-\frac{q}{n_1}\right)^2+y^2+z^2 \text{ etc.}$$

or
$$\frac{q^2}{{n_1}^2}=2x.\frac{q}{n_1} \quad \text{or} \quad x=\frac{q}{2n_1}.$$

Similarly, $\qquad y=\dfrac{q}{2n_2}, \quad z=\dfrac{q}{2n_3}.$

$$\therefore \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{2}{q}(an_1+bn_2+cn_3)=\frac{2}{q}\cdot q=2.$$

Ex. 6. *If any tangent plane to the sphere* $x^2+y^2+z^2=d^2$ *makes intercepts a, b, c on the axes, prove by vectors that*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{d^2}.$$

[See Q. 5 P. 182 of Author's Solid Geometry].

The centre of the sphere is origin and radius is d and hence its vector equation is $\mathbf{r}^2=d^2$.

Any plane $\mathbf{r}\cdot\mathbf{n}=q$ will be a tangent plane if perpendicular from centre is equal to radius

i.e., $$\frac{q}{|\mathbf{n}|}=d \quad \text{or} \quad q=nd.$$

The intercepts made by the plane on the axes are

$$\frac{q}{\mathbf{i}\cdot\mathbf{n}}=a, \quad \frac{q}{\mathbf{j}\cdot\mathbf{n}}=b, \quad \frac{q}{\mathbf{k}\cdot\mathbf{n}}=c.$$

$$\therefore \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{q^2}\left[(\mathbf{i}\cdot\mathbf{n})^2+(\mathbf{j}\cdot\mathbf{n})^2+(\mathbf{k}\cdot\mathbf{n})^2\right]$$

$$=\frac{1}{n^2a^2}[n^2\cos^2\alpha+n^2\cos^2\beta+n^2\cos^2\gamma]$$

$$=\frac{1}{d^2}\quad\because\quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$

α, β, γ being the angles which the normal makes with the axes.

Ex. 7. *The plane through the intersection of two spheres is perpendicular to the line joining their centres.*

Ex. 8. *The mid. points of the six edges of a tetrahedron ABCD lies on a sphere of radius* $\mathbf{r}$; *then prove that :—*

(i) Centre of the sphere is the centroid of the tetrahedron.

(ii) The sum of the squares on the line joining the centre to the vertices of the tetrahedron is $12r^2$.

(iii) The sum of the squares on the pairs of opposite edges is $16r^2$ *and that these edges are perpendicular.*

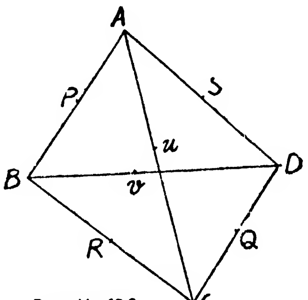


Fig No.100

Let the centre of the sphere be taken as origin and the position vector of the vertices be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively.

The mid. points of the edges (writen in groups of opposite edges) are

$$\frac{\mathbf{a}+\mathbf{b}}{2},\ \frac{\mathbf{c}+\mathbf{d}}{2};\quad \frac{\mathbf{b}+\mathbf{c}}{2},\ \frac{\mathbf{a}+\mathbf{d}}{2};\quad \frac{\mathbf{a}+\mathbf{c}}{2},\ \frac{\mathbf{b}+\mathbf{d}}{2}$$

i. e. P and Q *i. e.* R and S *i. e.* L and M

Since these points lie on the sphere whose centre is origin and radius $\mathbf{r}$, we have

$$\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2=\left(\frac{\mathbf{c}+\mathbf{d}}{2}\right)^2=\left(\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2=\left(\frac{\mathbf{a}+\mathbf{d}}{2}\right)^2=\left(\frac{\mathbf{a}+\mathbf{c}}{2}\right)^2$$

$$=\left(\frac{\mathbf{b}+\mathbf{d}}{2}\right)^2=r^2\ \ldots\quad\ldots\quad\ldots(1)$$

(1) We have to prove that centre of the sphere is the

centroid or we have to prove that $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}=0$ as we have taken the centre at the origin.

Now $$(\mathbf{a}+\mathbf{b})^2=(\mathbf{c}+\mathbf{d})^2=4r^2$$

or $$(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}) \cdot (\mathbf{a}+\mathbf{b}-\mathbf{c}-\mathbf{d})=0$$

or $$(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}) \cdot \tfrac{1}{2}\overrightarrow{QP}=0.$$

Similarly,

$$(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}) \cdot \tfrac{1}{2}\overrightarrow{SR}=0 \text{ and } (\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}) \cdot \tfrac{1}{2}\overrightarrow{ML}=0.$$

Above relations show that $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}$ is perpendicular to $\overrightarrow{QP}$, $\overrightarrow{SR}$ and $\overrightarrow{ML}$ which are the joins of the middle points of pairs of opposite edges and are non-coplanar vectors. Now if a vector is perpendicular to three non-coplanar vectors, then it should be a zero vector.

$$\therefore \quad \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0.$$

(2) Now have to prove that

$$OA^2+OB^2+OC^2+OD^2=12r^2$$

or $$\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+\mathbf{d}^2=12r^2.$$

Adding the relations in (1), we get

$$\Sigma\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2=6r^2 \quad \text{or} \quad 3\Sigma\mathbf{a}^2+2\Sigma\mathbf{a}\cdot\mathbf{b}=24r^2 \qquad \ldots(2)$$

Now $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$.

Squaring, we get $\Sigma\mathbf{a}^2+2\Sigma\mathbf{a}\cdot\mathbf{b}=0$

or $$-\Sigma\mathbf{a}^2=2\Sigma\mathbf{a}\cdot\mathbf{b}.$$

$\therefore \quad 3\Sigma\mathbf{a}^2-\Sigma\mathbf{a}^2=24r^2 \quad$ or $\quad \Sigma\mathbf{a}^2=12r^2$ from (2).

(3) The sum of the squares on the opposite edges is

$$AB^2+CD^2=(\mathbf{b}-\mathbf{a})^2+(\mathbf{d}-\mathbf{c})^2=\Sigma\mathbf{a}^2-2(\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{d})$$
$$=12r^2-2(\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{d}) \qquad \ldots \quad \ldots \quad \ldots(3)$$

Now from (1), $\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^2+\left(\frac{\mathbf{c}+\mathbf{d}}{2}\right)^2=r^2+r^2$

or $$\Sigma\mathbf{a}^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{d})=8r^2.$$

$$\therefore \quad 2\,(\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{d})=8r^2-\Sigma\mathbf{a}^2=8r^2-12r^2=-4r^2.$$

$$\therefore \quad AB^2+CD^2=12r^2-(-4r^2)=16r^2.$$

Again $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{d}=-2r^2=\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{d}$

$$\mathbf{b}\cdot(\mathbf{a}-\mathbf{c})-\mathbf{d}\cdot(\mathbf{a}-\mathbf{c})=0 \quad \text{or} \quad (\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{d})=0$$

i. e. $\overrightarrow{CA}\perp\overrightarrow{DB}.$ Hence proved.

Ex. 9. *Prove that any straight line drawn from a point O to intersect a sphere is cut harmonically by the surface and the polar plane of O.* **(Agra 53, 60)**

Let the equation to the sphere be $\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k=0$ and the point O be taken as origin. Therefore any line through O is $\mathbf{r}=t\,\hat{\mathbf{b}}$ where t stands for the distance of any point on it from O. Its points of intersection with the sphere are given by

$$t^2-2\hat{\mathbf{b}}\cdot\mathbf{c}\,t+k=0; \quad \because \quad \hat{\mathbf{b}}^2=1. \quad \textbf{[§ 13 P. 195]}$$

If it meets the sphere in P and Q, then OP and OQ are the two values of t given by above

$$\therefore \quad \frac{1}{OP}+\frac{1}{OQ}=\frac{1}{t_1}+\frac{1}{t_2}=\frac{t_1+t_2}{t_1t_2}=\frac{2\hat{\mathbf{b}}\cdot\mathbf{c}}{k}\ldots \quad \ldots(1)$$

Again polar plane of O w.r.t. the sphere is $\mathbf{r}\cdot\mathbf{c}=k.$

[§ 14 P. 200]

Again if $\mathbf{r}=t\hat{\mathbf{b}}$ cuts this plane in R, then $t\hat{\mathbf{b}}\cdot\mathbf{c}=k$ where t now stands for OR.

$$\therefore \quad \frac{2}{OR}=\frac{2}{t}=\frac{2\hat{\mathbf{b}}\cdot\mathbf{c}}{k}=\frac{1}{OP}+\frac{1}{OQ} \text{ [from (1)].}$$

$$\therefore \quad \frac{1}{OP},\ \frac{1}{OR},\ \frac{1}{OQ} \text{ are in A.P.}$$

or OP, OR, OQ are in H.P.

Hence the line is cut harmonically by the surface and the polar plane of O.

§ 16. Volume of a tetrahedron.

Let the position vectors of the three coterminous edges OA, OB and OC of the tetrahedron $OABC$ be **a**, **b**, **c** respectively with respect to origin.

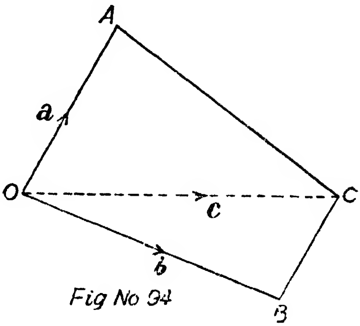


Fig No 94

Now we know that volume of a tetrahedron is $\frac{1}{3}$ area of base $OBC \times$ height of A from the plane of base.

Area of $\overrightarrow{\triangle OBC}$ $\frac{1}{2}\mathbf{b}\times\mathbf{c}$ **(Ex. 6, P. 128)** which represents a vector perpendicular to the plane of $\triangle OBC$.

$$\therefore \quad \text{volume of tetrahedron} = \tfrac{1}{3} \cdot (\tfrac{1}{2}\mathbf{b}\times\mathbf{c}) \cdot \mathbf{a}$$
$$= \tfrac{1}{6}(\mathbf{b}\times\mathbf{c}) \cdot \mathbf{a} = \tfrac{1}{6}[\mathbf{abc}].$$

Again we know from **§ 5, P. 130** that the volume of a parallelopiped whose three coterminous edges are **a**, **b**, **c** is [**abc**].

$\therefore$ **volume of tetrahedron** $= \frac{1}{6}$ **volume of parallelopiped.**

Cor. 1. *Volume of tetrahedron in terms of position vectors of the four vertices, neither of which is at origin.*

We have seen that when one of the vertices is at the origin, the volume of the tetrahedron is

$$\tfrac{1}{6}[\mathbf{abc}] = \tfrac{1}{6}\mathbf{a} \cdot (\mathbf{b}\times\mathbf{c})$$
$$= \tfrac{1}{6}\overrightarrow{OA} \cdot (\overrightarrow{OB}\times\overrightarrow{OC}).$$

Let the position vectors of the four points A, B, C and

O be **a**, **b**, **c** and **d** with respect to any origin O'.

(Agra 39, 51)

$$\therefore \quad \overrightarrow{OA}=\overrightarrow{O'A}-\overrightarrow{O'O}=\mathbf{a}-\mathbf{d}.$$

Similarly $\overrightarrow{OB}=\mathbf{b}-\mathbf{d}$ and $\overrightarrow{OC}=\mathbf{c}-\mathbf{d}$.

$\therefore$ volume of tetrahedron

$$=\tfrac{1}{6}\cdot[\mathbf{a}-\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{c}-\mathbf{d}]$$

$$=\tfrac{1}{6}\cdot(\mathbf{a}-\mathbf{d})\cdot[(\mathbf{b}-\mathbf{d})\times(\mathbf{c}-\mathbf{d})].$$

$$V=\tfrac{1}{6}\,[(\mathbf{a}-\mathbf{d})\cdot\{\mathbf{b}\times\mathbf{c}-\mathbf{d}\times\mathbf{c}-\mathbf{b}\times\mathbf{d}\}], \quad \because \ \mathbf{d}\times\mathbf{d}=0$$

$$=\tfrac{1}{6}\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})-\mathbf{a}\cdot(\mathbf{d}\times\mathbf{c})-\mathbf{a}\cdot(\mathbf{b}\times\mathbf{d})-\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})].$$

Scalar triple product is zero when two vectors are equal. **(Cor. 2, P. 137)**

$$=\tfrac{1}{6}\,\{[\mathbf{abc}]-[\mathbf{abd}]+[\mathbf{acd}]-[\mathbf{bcd}]\}.$$

$$\because \quad -\mathbf{a}\cdot(\mathbf{d}\times\mathbf{c})=\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})=[\mathbf{acd}].$$

Rule :—The above form is quite convenient to remember ; *i.e.* write **a**, **b**, **c**, **d** and strike one letter from the end and then the next and so on. Then form the scalar triple product of the three vectors thus left and connect them with alternately +ive and −ive signs.

Cor. 2. Condition for any four points to be coplanar.

In case the four points are coplanar, then the volume of the tetrahedron should be zero or otherwise $\mathbf{a}-\mathbf{d}$, $\mathbf{b}-\mathbf{d}$, $\mathbf{c}-\mathbf{d}$ are coplanar, *i.e.* $[\mathbf{a}-\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{c}-\mathbf{d}]=0$ which when expanded reduces to

$$[\mathbf{abc}]-[\mathbf{abd}]+[\mathbf{acd}]-[\mathbf{bcd}]=0.$$

Cor. 3. Volume of tetrahedron in terms of the coordinates of the vertices. **(Agra 48)**

Let (x_r, y_r, z_r), where $r=1, 2, 3, 4$ be the coordinates of the four vertices of the tetrahedron so that the position vectors of the four points in terms of unit vectors **i**, **j** and **k** are

$$\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_2\mathbf{k} \text{ etc.,}$$

$$\mathbf{d}=x_4\mathbf{i}+y_4\mathbf{j}+z_4\mathbf{k},$$

so that $\mathbf{a}-\mathbf{d}=(x_1-x_4)\,\mathbf{i}\times(y_1-y_4)\,\mathbf{j}+(z_1-z_4)\,\mathbf{k}.$

Now the volume of the tetrahedron is

$$\tfrac{1}{6}\,[\mathbf{a}-\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{c}-\mathbf{d}]$$

or $$V=\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}$$ **[Cor. 4 P. 138]**

The value of this determinant remains unchanged if we make it of fourth order by adding one row of 1, x_4, y_4, z_4 and one column of 1, 0, 0, 0 as shown below

$$\therefore\quad V=\frac{1}{6}\begin{vmatrix} 1 & x_4 & y_4 & z_4 \\ 0 & x_1-x_4 & y_1-y_4 & z_1-z_4 \\ 0 & x_2-x_4 & y_2-y_4 & z_2-z_4 \\ 0 & x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}.$$

Adding the elements of 1st row to 2nd, 3rd and 4th rows, we get

$$V=\frac{1}{6}\begin{vmatrix} 1 & x_4 & y_4 & z_4 \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix}.$$

Exercise

Ex. 1. *Prove that the volume of a tetrahedron bounded by the four planes* $\mathbf{r}\cdot(m\mathbf{j}+n\mathbf{k})=0$, $\mathbf{r}\cdot(n\mathbf{k}+l\mathbf{i})=0$, $\mathbf{r}\cdot(l\mathbf{i}+m\mathbf{j})=0$ *and* $\mathbf{r}\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=p$ *is* $\dfrac{2p^3}{3lmn}$.

[Q. 7 P. 80 of Author's Solid Geometry]

(Agra 39, 45, 59; Benaras 54; Lucknow 52)

Let us find the points of intersection of the four planes taken three at a time. The cartesian equations of the planes are (i) $my+nz=0$, (ii) $nz+lx=0$, (iii) $lx+my=0$ and (iv) $lx+my+nz=p$.

Clearly the first three intersect at origin. Now let us find the point of intersection of (i), (ii) and (iv). Let it be (x, y, z) so that the position vector of this point is $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ (say).

Adding (i) and (ii), we get

$$lx+my+nz=0$$

or $$p+nz=0 \quad [\text{from (iv)}]; \quad \therefore \; z=-p/n.$$

$$\therefore \quad my=p \quad \text{or} \quad y=p/m \quad [\text{from (i)}]$$

and $$lx=p \quad \text{or} \quad x=p/l \quad [\text{from (ii)}].$$

Hence $\frac{p}{l}\mathbf{i}+\frac{p}{m}\mathbf{j}-\frac{p}{n}\mathbf{k}=\mathbf{c}$ (say).

Similarly points of intersection of (i), (iii), (iv) and (ii) (iii), (iv) are

$$\frac{p}{l}\mathbf{i}-\frac{p}{m}\mathbf{j}+\frac{p}{n}\mathbf{k}=\mathbf{b} \text{ (say)}$$

and $$\frac{-p}{l}\mathbf{i}+\frac{p}{m}\mathbf{j}+\frac{p}{n}\mathbf{k}=\mathbf{a} \text{ (say)}$$

Now the volume of a tetrahedron whose one vertex is at the origin is $\frac{1}{6}[\mathbf{abc}]$

$$=\frac{1}{6}\begin{vmatrix} -p/l & p/m & p/n \\ p/l & -p/m & p/n \\ p/l & p/m & -p/n \end{vmatrix}=\frac{p^3}{6lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=\frac{p^3}{6lmn}\{-1\,(1-1)-1\,(-1-1)+1\,(1+1)\}=\frac{4}{6}\frac{p^3}{lmn}=\frac{2}{3}\frac{p^3}{lmn}.$$

Ex. 2. *Prove the following formula for the volume V of a tetrahedron in terms of lengths of three concurrent edges and their mutual inclinations.* **[Refer Author's Solid Geometry P. 75]**

$$V^2=\frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\phi & \cos\psi \\ \cos\phi & 1 & \cos\theta \\ \cos\psi & \cos\theta & 1 \end{vmatrix}$$

(Agra 57, Luck. 55)

Let the three concurent edges concur at origin O and the position vectors of the other vertices A, B, C be

$$\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k},\ \mathbf{b}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k},\ \mathbf{c}=x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}.$$

$\therefore \quad OA^2=x_1^2+y_1^2+z_1^2=a^2,\ OB^2=\Sigma x_2^2,\ OC^2=\Sigma x_3^2.$

Again $\mathbf{a}\cdot\mathbf{b}=x_1x_2+y_1y_2+z_1z_2=ab\cos\phi$

$\mathbf{b}\cdot\mathbf{c}=\Sigma x_2x_3=bc\cos\theta,$

$\mathbf{c}\cdot\mathbf{a}=\Sigma x_3x_1=ca\cos\psi.$

$$V=\tfrac{1}{6}[\mathbf{abc}]=\tfrac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$\therefore \quad V^2=\tfrac{1}{36}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\times\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$=\tfrac{1}{36}\begin{vmatrix} \Sigma x_1^2 & \Sigma x_1x_2 & \Sigma x_1x_3 \\ \Sigma x_1x_2 & \Sigma x_2^2 & \Sigma x_2x_3 \\ \Sigma x_1x_3 & \Sigma x_2x_3 & \Sigma x_3^2 \end{vmatrix}$$

$$=\tfrac{1}{36}\begin{vmatrix} a^2 & ab\cos\phi & ac\cos\psi \\ ab\cos\phi & b^2 & bc\cos\theta \\ ac\cos\psi & bc\cos\theta & c^2 \end{vmatrix}$$

$$=\frac{abc}{36}\begin{vmatrix} a & b\cos\phi & c\cos\psi \\ a\cos\phi & b & c\cos\theta \\ a\cos\psi & b\cos\theta & c \end{vmatrix}$$

$$=\frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\phi & \cos\psi \\ \cos\phi & 1 & \cos\theta \\ \cos\psi & \cos\theta & 1 \end{vmatrix}.$$

Ex. 3. *G_1, G_2, G_3 are the centroids of the triangular faces OBC, OCA, OAB of a tetrahedron $OABC$. Prove that the volume of the tetrahedron $OABC$ is to the volume of the parallelopiped constructed with OG_1, OG_2 and OG_3 as coterminous edges as 9 : 4.*

Ex. 4. *Prove that each of the four faces of a terahedron subtends the same volume at the centroid.*

Let G the centroid of **a**, **b**, **c** and **d** be taken as origin.

$$\therefore \quad \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0 \quad \dots \quad \dots \quad \dots(1)$$

Volume of tetrahedron $GABC$, G being origin$=\frac{1}{6}[\mathbf{abc}]$.

Volume of tetrahedron $GBCD=\frac{1}{6}[\mathbf{bcd}]$

$$=\tfrac{1}{6}\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})=\tfrac{1}{6}\mathbf{b}\cdot\{\mathbf{c}\times(-\mathbf{a}-\mathbf{b}-\mathbf{c})\} \text{ from (1)}$$
$$=\tfrac{1}{6}\mathbf{b}\cdot\{-\mathbf{c}\times\mathbf{a}-\mathbf{c}\times\mathbf{b}\}=-\tfrac{1}{6}\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$$
$$=\tfrac{1}{6}\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=0$$
$$=\tfrac{1}{6}[\mathbf{abc}].$$

Ex. 5. *In tetrahedron $OABC$ prove that the volume V is given by the formula $\frac{1}{6}\,AB\cdot OC\cdot p\sin\theta$ where p is the shortest distance between AB and OC.*

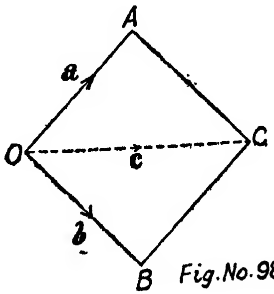


Fig.No.98

From the figure we observe that $\overrightarrow{AB}$ is parallel to $\mathbf{b}-\mathbf{a}$ and OC is parallel to **c**.

$\therefore$ shortest distance is parallel to $(\mathbf{b}-\mathbf{a})\times\mathbf{c}$.

Also **a** is a point on OA and **c** is a point on OC.

$\therefore$ shortest distance is projec-

tion of $\mathbf{a}-\mathbf{c}$ on $(\mathbf{b}-\mathbf{a})\times\mathbf{c}$. [§ 11 P. 177]

If p be the shortest distance, then

$$p=\frac{(\mathbf{a}-\mathbf{c})\cdot[(\mathbf{b}-\mathbf{a})\times\mathbf{c}]}{|(\mathbf{b}-\mathbf{a})\times\mathbf{c}|}$$

$$=\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c})}{AB\cdot OC\sin\theta}$$

$$=\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{AB\cdot OC\sin\theta},$$

$\because$ scalar triple product vanishes when two vectors are equal.

$$\therefore\quad p=\frac{[\mathbf{abc}]}{AB\cdot OC\sin\theta};$$

$$\therefore\quad V=\tfrac{1}{6}[\mathbf{abc}]=\tfrac{1}{6}AB\cdot OC\cdot p\sin\theta.$$

Ex. 6. *Show that the volume of a pyramid of which the vertex is a given point (x, y, z) and the base a triangle formed by joining the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ in rectangular coordinates is*

$$\tfrac{1}{6}abc\left[\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right].$$ **(Agra 47)**

We know that a triangular pyramid is a tetrahedron. In terms of the unit vectors the given points are $a\mathbf{i}$, $b\mathbf{j}$, $c\mathbf{k}$ and $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, say vectors A, B, C and D.

$$V=\tfrac{1}{6}\,|\,A-D\quad B-D\quad C-D\,|$$ [Cor. 1 P. 188]

$$A-D=(a-x)\,\mathbf{i}-y\mathbf{j}-z\mathbf{k},$$

$$B-D=-x\mathbf{i}+(b-y)\,\mathbf{j}-z\mathbf{k},$$

$$C-D=-x\mathbf{i}-y\mathbf{j}+(c-z)\,\mathbf{k};$$

$$\therefore\quad V=\tfrac{1}{6}\begin{vmatrix} a-x & -y & -z \\ -x & b-y & -z \\ -x & -y & c-z \end{vmatrix}$$

$$=\tfrac{1}{6}abc\begin{vmatrix} 1-x/a & -y/b & -z/c \\ -x/a & 1-y/b & -z/c \\ -x/a & -y/b & 1-z/c \end{vmatrix}.$$

Adding column nos. 1, 2, and 3, we get

$$V=\tfrac{1}{6}abc\,(1-x/a-y/b-z/c)\begin{vmatrix} 1 & -y/b & -z/c \\ 1 & 1-y/b & -z/c \\ 1 & -y/b & 1-z/c \end{vmatrix}$$

$$=\tfrac{1}{6}\,abc\,(1-x/a-y/b-z/c)\begin{vmatrix} 1 & -y/b & -z/c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

(by R_2-R_1 and R_3-R_1)

$$=\tfrac{1}{6}\,abc\,(1-x/a-y/b-z/c)$$

$$=\tfrac{1}{6}abc\,(x/a+y/b+z/c-1).$$

Ex. 7. *G is the centroid of the tetrahedron OABC ; O′A′B′C′ is another tetrahedron such that OO′, AA′, BB′ and CC′ are all bisected at G ; show that G is also the centroid of the tetrahedron* **O′A′B′C′**.

AGRA UNIVERSITY SOLVED PAPERS

1952

1. Find the vector equation of a straight line passing through two given points. Prove by vectorial methods that the following are concurent :—

(a) the bisectors of the angles of a triangle ; and (b) the medians of a triangle.

(a) Cor. 2 P. 48. (b) Ex. 1 P. 52, Ex. 2 P. 53.

2. (a) Find the vector equation of a sphere.

(b) (i) Show that any diameter of a sphere subtends a right angle at a point on the surface.

(ii) Prove that if a point is equidistant from the vertices of a right-angled triangle, its join to the middle point of the hypetenuse is perpendicular to the plane of the triangle.

(a) § 12 P. 191.

(b) (i) See general method P. 194. (ii)Q. 11 (b) P. 111.

3. (a) Obtain the equation of a straight line perpendicular to two non-intersecting lines.

(b) Prove that the locus of the middle points of all straight lines terminated by two fixed non-intersecting straight lines is a plane bisecting their common perpendicular at right-angles.

(a) § 11 P. 177.

1953

1. (a) Define 'Centroid'. Show that the centroid is independent of the origin of vectors.

(b) Prove that the lines joining the vertices of a tetrahedron to the centroids of area of the opposite faces are concurrent.

(a) § 2 P. 37, § 4 P. 40.

(b) Ex. 6. P. 59.

2. (a) In a tetrahedron, if two pairs of opposite edges are perpendicular, prove that the third pair are also perpendicular to each other, and the sum of the squares on two opposite edges is the same for each pair.

(b) Prove that any straight line drawn from a point *O* to intersect a sphere is cut harmonically by the surface and the polar plane of *O*.

2. (a) Ex. 1 P. 104, (b) Ex. 19 P. 208.

3. Establish the following relations :—

(i) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0.$

(ii) $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2.$

(iii) $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=0.$

(i) See bottom P. 142.

(ii) Cor. P. 146.

(iii) Q. 11 P. 154.

1954

1. Prove the following by vector methods :—

(a) The internal bisectors of the angles of a triangle are concurrent.

(b) *ABCD* is a parallelogram and *O* the point of intersection of the diagonals. Show that for any origin (not necessarily in the plane of the figure) the sum of the position vectors of the vertices is equal to four times that of *O*.

(a) Ex. 2 P. 53, (b) Just as Q. 15 (b) P. 30.

2. Give vectorial solutions of the following :—

(a) The area of the triangle formed by joining the middle point of one of the non-parallel sides of a trapezium to the extremities of the opposite side is half that of the trapezium.

(b) Find the coordinates of the centre of the sphere inscribed in the tetrahedron bounded by the planes

$$\mathbf{r}\cdot\mathbf{i}=0,\ \mathbf{r}\cdot\mathbf{j}=0,\ \mathbf{r}\cdot\mathbf{k}=0 \text{ and } \mathbf{r}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=\mathbf{a}.$$

(a) Ex. 12 P. 132, (b) Ex. 1 P. 198.

3. Establish the following vector relations :—

(i) $\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=\mathbf{b}\cdot\mathbf{d}\ \mathbf{a}\times\mathbf{c}-\mathbf{b}\cdot\mathbf{c}\ \mathbf{a}\times\mathbf{d}.$

(ii) $[\mathbf{lmn}][\mathbf{abc}]=\begin{vmatrix} \mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\cdot\mathbf{c} \\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\cdot\mathbf{c} \\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\cdot\mathbf{c} \end{vmatrix}$

Q. 10 P. 154, (b) Ex. 2 P. 147.

1955

1. Prove the following by vector methods —:

(a) The medians of a triangle are concurrent.

(b) The four diagonals of a parallelopiped, and the joins of the mid. points of opposite edges, are concurrent at a common point of bisection.

(c) The three points whose position vectors are **a**, **b**, $3\mathbf{a}-2\mathbf{b}$ are collinear.

(a) Ex. 1 P. 52, (b) Ex. 7 P. 61.

(c) Q. 26, (b) P. 77.

2. (a) Find by vector method, the equation of the line of intersection of two planes.

(b) Show that the line of intersection of

$$\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=0 \text{ and } \mathbf{r}\cdot(3\mathbf{i}+2\mathbf{j}+\mathbf{k})=0$$

is equally inclined to **i** and **k**, and makes an angle $\frac{1}{2}\sec^{-1}3$ with **j**.

(a) § 8 P. 171, (b) Ex. 5 P. 182.

3. (a) Prove the relation

$$\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=\mathbf{b}\cdot\mathbf{d}\ \mathbf{a}\times\mathbf{c}-\mathbf{b}\cdot\mathbf{c}\ \mathbf{a}\times\mathbf{d},$$

and hence expand

$$\mathbf{a}\times[\mathbf{b}\times\{\mathbf{c}\times(\mathbf{d}\times\mathbf{e})\}].$$

(b) Find the equation of the straight line through the point **c**, intersecting both the lines

$$\mathbf{r}-\mathbf{a}=s\mathbf{b} \text{ and } \mathbf{r}-\mathbf{a}'=t\mathbf{b}'.$$

(a) Q. 10 P. 154, (b) Ex. 14 P. 184.

1956

1. (a) Prove the following by vector methods :—

(i) The internal bisector of the angle A of a triangle ABC divides the side BC in the ratio $AB : AC$.

(ii) The join of the mid. points of two sides of a triangle is parallel to the third side, and of half its length.

(b) What is the vector equation of the straight line through the points $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $3\mathbf{k}-2\mathbf{j}$?

Find where the line cuts the plane through the origin and the points $4\mathbf{j}$ and $2\mathbf{i}+\mathbf{k}$.

(a) (i) Ex. 3 P. 57, (ii) Ex. 19 P. 72.

(b) Q. 2 P. 90.

2. (a) Find the equations of the planes bisecting the angles between the two given planes

$$\mathbf{r}\cdot\mathbf{n}=q \text{ and } \mathbf{r}\cdot\mathbf{n}'=q'.$$

(b) Find the coordinates of the centre of the sphere inscribed in the tetrahedron bounded by the planes

$$\mathbf{r}\cdot\mathbf{i}=0,\ \mathbf{r}\cdot\mathbf{j}=0,\ \mathbf{r}\cdot\mathbf{k}=0 \text{ and } \mathbf{r}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=a.$$

(a) § 6 P. 169, (b) Ex. 1 P. 198.

3. (a) Prove that

$$[\mathbf{lmn}][\mathbf{abc}]=\begin{vmatrix} \mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\cdot\mathbf{c} \\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\cdot\mathbf{c} \\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\cdot\mathbf{c} \end{vmatrix}.$$

(b) Prove the formula

$$\begin{aligned}[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}] &= [\mathbf{abd}]\,[\mathbf{cef}]-[\mathbf{abc}]\,[\mathbf{def}] \\ &= [\mathbf{abe}]\,[\mathbf{fcd}]-[\mathbf{abf}]\,[\mathbf{ecd}] \\ &= [\mathbf{cda}]\,[\mathbf{bef}]-[\mathbf{cdb}]\,[\mathbf{aef}].\end{aligned}$$

(a) Ex. 2 P. 148, (b) Ex. 6 P. 153.

1957

1. Prove the following by vector methods :—

(a) The internal bisectors of the angles of a triangle are concurrent.

(b) The area of the triangle formed by joining the mid. point of one of the non-parallel sides of a trapezium to the extremities of the opposite side is half that of the trapezium.

(a) Ex. 2 P. 53 ; (b) Q. 12 P. 132.

2. (a) Show that any given vector **r** can be expressed in terms of three given non-coplanar vectors α, β, γ in the form

$$\vec{\mathbf{r}}=\frac{[\mathbf{r}\beta\gamma]\,\vec{\alpha}+[\mathbf{r}\gamma\alpha]\,\vec{\beta}+[\mathbf{r}\alpha\beta]\,\vec{\gamma}}{[\alpha\beta\gamma]}.$$

(b) Prove that $[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2$ and express the result by means of determinants.

(a) § 8 P. 143-144, (b) Cor. P. 146.

3. (a) Explain the terms scalar and vector products of two vectors, giving illustrations.

(b) Prove the following formula for the volume V of a tetrahedron, in terms of the lengths of three concurrent edges and their mutual inclinations :

$$V^2=\frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\phi & \cos\psi \\ \cos\phi & 1 & \cos\theta \\ \cos\psi & \cos\theta & 1 \end{vmatrix}.$$

(a) § 2 P. 99 and § 3 P. 118, (b) Ex. 2 P. 212.

1958

1. (a) Define 'centroid'. Show that the centroid is independent of the origin of vectors.

(b) Two forces act at the corner A of a quadrilateral

$ABCD$, represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$; and two at C represented by $\overrightarrow{CB}$ and $\overrightarrow{CD}$. Show that their resultant is represented by $4\overrightarrow{PQ}$, where P, Q are the mid. points of AC, BD respectively.

(a) § 2 P. 37, § 4 P. 40.

(b) Q. 15 (a) P. 30.

2. (a) If any point O within a tetrahedron $ABCD$ is joined to the vertices, and AO, BO, CO, DO are produced to cut the opposite faces in P, Q, R, S respectively, then show that

$$\Sigma \frac{OP}{AP}=1.$$

(b) Prove that the three points whose position vectors are $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a}-2\mathbf{b}$ are collinear.

(a) Ex. 1 P. 89; (b) Q. 26 (b) P. 77.

3. (a) Define (1) the scalar, (2) the vector product of two vectors, and give instances of their application to mechanics.

(b) Find the straight line, through the point $\mathbf{c}_1$ which is parallel to the plane $\mathbf{r}\cdot\mathbf{a}=0$, and intersects the line

$$\mathbf{r}-\mathbf{a}'=t\mathbf{b}.$$

§ 2 P. 98 and § 3 P. 119 and see Q. 14 (b) P. 185.

1957

1. Prove bo vector method the following :—

(a) If a line be drawn parallel to the base of a triangle, the line which joins the opposite vertex to the intersection of the diagonals of the trapezoid thus formed bisects the base.

(b) The points $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $3(\mathbf{i}+\mathbf{j}-\mathbf{k})$ are equidistant from the plane $\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j}-7\mathbf{k})=0$, and are on oppposite sides of it.

(a) Ex. 22 P. 72. (b) § 5 P. 166 and Ex. 3. (b) P. 168.

2. Prove that :—

$$(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=\mathbf{A}\,.\,\mathbf{BC}-\mathbf{A}\,.\,\mathbf{CB},$$

and that the volume of the tetrahedron bounded by the four planes

$$\mathbf{r}\,.\,(m\mathbf{j}+n\mathbf{k})=0,\ \mathbf{r}\,.\,(n\mathbf{k}+l\mathbf{i})=0,$$
$$\mathbf{r}\,.\,(l\mathbf{i}+m\mathbf{j})=0 \text{ and } \mathbf{r}\,.\,(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=p$$

is
$$2p^3/3lmn.$$

§ 6 P. 140, Ex. 1 P. 211.

3. (a) Prove that the shortest distance between the two opposite edges of a regular tetrahedron is equal to half the diagonal of the square described on an edge.

(b) Establish the vector formula :

$$[\mathbf{a}\times\mathbf{p}\ \mathbf{b}\times\mathbf{q}\ \mathbf{c}\times\mathbf{r}]+[\mathbf{a}\times\mathbf{q}\ \mathbf{b}\times\mathbf{r}\ \mathbf{c}\times\mathbf{p}]$$
$$+[\mathbf{a}\times\mathbf{r}\ \mathbf{b}\times\mathbf{p}\ \mathbf{c}\times\mathbf{q}]=0.$$

(a) Ex. 22 (d) P. 188. (b) Ex. 7 P. 153.

1960

1. (a) Find the vector equation to a sphere. Prove also that any straight line drawn from a point O to intersect a sphere is cut harmonically by the surface and the polar plane of O.

(b) Find the equation of the straight line through the point **d** and equally inclined to the vectors **a**, **b**, **c** in the form

$$\mathbf{r}=\mathbf{d}+s\left(\frac{\mathbf{a}}{a}+\frac{\mathbf{b}}{b}+\frac{\mathbf{c}}{c}\right)$$

and
$$\mathbf{r}=\mathbf{d}+s\left[\frac{a\,(\mathbf{b}\times\mathbf{c})+b\,(\mathbf{c}\times\mathbf{a})+c\,(\mathbf{a}\times\mathbf{b})}{abc}\right].$$

(a) § 12 P. 191, Ex. 19 P. 208, (b) Ex. 4 P. 124.

2. (a) What do you understand by a system of reciprocal vectors ? Show that any given vector r can be expressed in terms of three given non-coplanar vectors α, β, γ in the form

$$\mathbf{r}=\frac{[\mathbf{r}\beta\gamma]\,\alpha+[\mathbf{r}\gamma\alpha]\,\beta+[\mathbf{r}\alpha\beta]\,\gamma}{[\alpha\beta\gamma]}.$$

(b) Prove that $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2$ and express the result by means of determinants.

(a) § 9 P. 144, (b) Cor. P. 146.

3. (a) Prove the formula

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=0$$

and use it to show that

$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B.$$

(b) Prove that the shortest distances between a diagonal of a rectangular parallelopiped whose sides are **a, b, c** and the edges not meeting it, are

$$\frac{bc}{\sqrt{(b^2+c^2)}},\ \frac{ca}{\sqrt{(c^2+a^2)}},\ \frac{ab}{\sqrt{(a^2+b^2)}}.$$

(a) Ex. 11 P. 154, (b) Ex. 7 P. 180.

1961

1. Prove by vector methods the following :—

(a) If any point O within a tetrahedron $ABCD$ is joined to the vertices, and AO, BO, CO, DO are produced to cut the opposite faces in P, Q, R, S respectively, then prove that $\Sigma\frac{OP}{AP}=1$.

(b) If through any point within a triangle, lines be drawn parallel to the sides, show that the sum of the ratios of these lines to their corresponding sides is 2.

Ex. 1 P. 89, Ex. 22 (b), P, 73.

2. (a) Show that the line of intersection of

$$\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=0 \text{ and } \mathbf{r}\cdot(3\mathbf{i}+2\mathbf{j}+\mathbf{k})=0$$

is equally inclined to $\mathbf{i}$ and $\mathbf{k}$ and makes an angle $\frac{1}{2}\sec^{-1}3$ with $\mathbf{j}$.

(b) Find by vector method, the equation to the line of intersection of two given planes.

(a) Ex. 5 P. 182, (b) § 8 P. 171.

3. (a) Find the equation of the straight line through the point $\mathbf{c}$, intersecting both the lines

$$\mathbf{r}-\mathbf{a}=s\mathbf{b} \text{ and } \mathbf{r}-\mathbf{a}'=t\mathbf{b}'.$$

(b) Prove the formulae

$$[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}]=[\mathbf{abe}]\,[\mathbf{fcd}]-[\mathbf{abf}]\,[\mathbf{ecd}]$$
$$=[\mathbf{cda}]\,[\mathbf{bef}]-[\mathbf{cdb}]\,[\mathbf{aef}].$$

(a) Ex. 14 P. 184, (b) Ex. 6 P. 153.

RAJPUTANA UNIVERSITY SOLVED PAPERS

1959

1. Define scalar triple product and prove the following :—

(a) A cyclic permutation of three vectors does not change the value of the scalar triple product but an anti-cyclic permutation changes the value in sign but not in magnitude.

(b) The position of a dot and cross can be interchanged without changing its value.

§ 4 and 5 page 134–136.

2. If **p**, **q**, **r** are three vectors defined by the relation

$$\mathbf{p}=\frac{\mathbf{b}\times\mathbf{c}}{(\mathbf{abc})},\quad \mathbf{q}=\frac{\mathbf{c}\times\mathbf{a}}{(\mathbf{abc})},\quad \mathbf{r}=\frac{\mathbf{a}\times\mathbf{b}}{(\mathbf{abc})},$$

where **a**, **b**, **c** are vectors and the scalar triple product $(\mathbf{abc})\neq 0$, prove that $(\mathbf{abc})\,(\mathbf{pqr})=1$ and obtain the values of **a**, **b** and **c** in terms of **p**, **q** and **r**.

§ 9 and Property 3 page 144–145.

1960

100% of the questions were set from this book.

1. (a) Show that every vector can be represented as a linear combination of two non-collinear vectors coplanar with the original vector.

(b) Show by the method of vectors that the straight line joining the middle points of two sides of a triangle is parallel to the third side and is of half its length.

(a) § 6 page 14, (b) Ex. 19 page 72.

PUNJAB UNIVERSITY SOLVED PAPERS

1960

100% of the questions were set from this book.

1. (a) The necessary and sufficient condition for four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, to be coplanar is that there exist four scalars x, y, z, t, not all zero, such that

$$x\,\vec{a}+y\,\vec{b}+z\,\vec{c}+t\,\vec{d}=0,\quad x+y+z+t=0.$$

(b) Show that the internal bisector of any angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(c) Show that the external bisectors of the three plane angles of each trihedron of a given tetrahedron are coplanar.

(a) § 10 page 86, (b) Ex. 3 page 57.

2. (a) Define the vector product of two vectors and prove that

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\,.\,\vec{c}\right)\vec{b}-\left(\vec{a}\,.\,\vec{b}\right)\vec{c}.$$

(b) Find the condition that

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$$

and interpret it geometrically.

(c) Prove that

$$\left[\vec{b}\times\vec{c},\ \vec{c}\times\vec{a},\ \vec{a}\times\vec{b}\right]=\left[\vec{a},\ \vec{b},\ \vec{c},\right]^2.$$

$$\vec{a}.\vec{b}\ \vec{c}+\vec{b}.\vec{c}\ \ \vec{a}=2\vec{c}\ .\ \vec{a}\ \vec{b},$$

(a) § 3 P. 118, § 6 P. 140, (b) Expand both sides by (a), (c) Cor. P. 146.

3. (a) Show that

$$\vec{a}.\left(\vec{b}+\vec{c}\right)=\vec{a}.\vec{b}+\vec{a}.\vec{c}.$$

(b) Find the distance of the point $\vec{a}$, from the plane $\vec{r}\,.\,\vec{n}=q$ measured parallel to the line $\vec{r}=\vec{b}+t\,\vec{c}$.

(c) Show that the middle point of the hypotenuse of a right-angled triangle is equidistant from its vertices.

(a) § 9 P. 100, (b) Cor. P. 168, (c) Ex. 10 P. 109.

4. (a) Show that in the scalar triple product, the dot and cross may be interchanged without changing the result.

(b) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be four vectors, express $\vec{d}$ as a linear combination of the three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$.

Hence prove that

$$\left[\vec{p}\;\vec{q}\;\vec{r}\right]\left[\vec{p'}\;\vec{q'}\;\vec{r'}\right]=\begin{vmatrix} \vec{p}\,.\,\vec{p'} & \vec{p}\,.\,\vec{q'} & \vec{p}\,.\,\vec{r'} \\ \vec{q}\,.\,\vec{p'} & \vec{q}\,.\,\vec{q'} & \vec{q}\,.\,\vec{r'} \\ \vec{r}\,.\,\vec{p'} & \vec{r}\,.\,\vec{q'} & \vec{r}\,.\,\vec{r'} \end{vmatrix},$$

where $\vec{p}, \vec{q'}\,\vec{r'};\,\vec{p'}\,\vec{q'}\,\vec{r'}$ are any vectors.

§ 5 P. 135, 136, (b) § 6 (iii) P. 16, (c) Ex. 2 P. 148.

5. (a) Show that the equation of the plane through two given points A, B with position vectors $\vec{a}, \vec{b}$ and parallel to a given vector $\vec{c}$ is

$$\vec{r}.\left(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right)=\left[\vec{a}\;\vec{b}\;\vec{c}\right].$$

(b) Show that the lines

$$\vec{r} \times \vec{a} = \vec{b} \times \vec{a}, \quad \vec{r} \times \vec{b} = \vec{a} \times \vec{b}$$

intersect and find the point of intersection.

(c) The position vectors of four points *A*, *B*, *C*, *D* relative to any origin *O* are denoted by $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Interpret geometrically the equations

(i) $\left(\vec{a} - \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \vec{0}$,

(ii) $\left(\vec{a} - \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = 0$.

(a) Result IV page 164, (b) Ex. 10 page 183, (c) Ex. 12 page 184.

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